

Optimal insurance demand under marked point processes shocks: a dynamic programming duality approach

Mohamed MNIF

LAMSIN

Ecole Nationale d'Ingénieurs de Tunis

B.P. 37, 1002, Tunis Belvédère, Tunisie

mohamed.mnif@enit.rnu.tn

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Abstract

We study the stochastic control problem of maximizing expected utility from terminal wealth under a non-bankruptcy constraint. The wealth process is subject to shocks produced by a general marked point process. The problem of the agent is to derive the optimal insurance strategy which allows "lowering" the level of the shocks. This optimization problem is related to a suitable dual stochastic control problem in which the delicate boundary constraints disappear. We characterize the dual value function as the unique viscosity solution of the corresponding a Hamilton Jacobi Bellman Variational Inequality (HJBVI in short).

Key words : Optimal insurance; stochastic control; duality; optional decomposition; dynamic programming principle; viscosity solution

MSC Classification (2000) : 93E20, 60J75, 49L25.

1 Introduction

We study the optimal insurance demand problem of an agent whose wealth is subject to shocks produced by some marked point process. Such a problem was formulated by Bryis [3] in continuous-time with Poisson shocks. Gollier [14] studied a similar problem where shocks are not proportional to wealth. An explicit solution to the problem is provided by Bryis by writing the associated Hamilton-Jacobi-Bellman (HJB in short) equation. In Bryis [3] and Gollier [14], they modeled the insurance premium by an affine function of the insurance strategy $\theta = (\theta_t)_{t \in [0, T]}$ which is the rate of insurance decided to be covered by the agent. If the agent is subject to some accident at time t which costs an amount Z , then he will pay $\theta_t Z$ and the insurance company reimburses the amount $(1 - \theta_t)Z$. They didn't assume any constraint on the insurance strategy which is not realistic.

In risk theory, Hipp and Plum [9] analysed the trading strategy, in risky assets, which is optimal with respect to the criterion of minimizing the ruin probability. They derived the HJB equation related to this problem and proved the existence of a solution and a verification theorem. When the claims are exponentially distributed, the ruin probability decreases exponentially and the optimal amount invested in risky assets converges to a constant independent of the reserve level. Hipp and Schmidli [10] have obtained the asymptotic behaviour of the ruin probability under the optimal investment strategy in the small claim case. Schmidli [22] studied the optimal proportional reinsurance policy which minimizes the ruin probability in infinite horizon. He derived the associated HJB equation, proved the existence of a solution and a verification theorem in the diffusion case. He proved that the ruin probability decreases exponentially whereas the optimal proportion to insure is constant. Moreover, he gave some conjecture in the Cramér-Lundberg case. Højgaard and Taksar [11] studied another problem of proportional reinsurance. They considered the issue of reinsurance optimal fraction, that maximizes the return function. They modelled the reserve process as a diffusion process.

Touzi [24] studied the problem of maximizing the expected utility from terminal wealth when the insurance strategy is valued in $[0, 1]$ at each time. He modeled the wealth process by a Doléans-Dade exponential process. He assumed a boundedness assumption on the jump term which guarantees the positivity of the wealth process. He solved this stochastic control problem by using duality method.

Duality method was introduced by Karatzas et al. [15] and Cox and Huang [5]. Cox and Huang characterized the optimal consumption- portfolio policies when there exist non-negativity constraints on consumption and on final wealth. They gave a verification theorem which involves a linear partial differential equation unlike the nonlinear Bellman equation. In few cases they constructed the optimal control. Extensions to the case of constrained investment are considered by Cvitanić and Karatzas [6] and to the case of incomplete markets by Karatzas et al. [16]. Typically, in incomplete markets, we have to solve a dual problem which leads in the Markov case to nonlinear partial differential equation.

In this paper, we model the claims by using a compound Poisson process. The insurance trading strategy is constrained to remain in $[0, 1]$. We impose a constraint of non-bankruptcy on the wealth process X_t of the agent for all t . The objective of the agent is to maximize the expected utility of the terminal wealth over all admissible strategies and to determine the optimal policy of insurance.

In our case the wealth process positivity constraint is a real one unlike the problem formulated in Touzi [24].

Our stochastic optimization problem is a particular case of a general structure of problems developed in Mnif and Pham [19] who considered the following optimization problem:

$$\max_{X \in \mathcal{X}_+(x)} E[U(X_T)], \quad x \in \mathbb{R}, \quad (1.1)$$

where $\mathcal{X}_+(x) := \{x + X : X \in \mathcal{X} \text{ s.t. } X_t \geq 0 \text{ for all } 0 \leq t \leq T\}$ where \mathcal{X} is a family of semi-martingales. Existence and uniqueness of solution of problem (1.1) is then proved. The optimal solution characterization is obtained from a dual formulation under minimal assumptions on the objective function.

In this paper, we study the dual value function by a PDE approach. The dual problem appears as a mixed control/singular optimization problem with dynamics $(Y_t = Z_t D_t, t \in [0, T])$ governed by a classical control term Z which comes from the insurance strategy and a singular term D which comes from the state constraint.

The originality of this paper is to study a stochastic control problem with state constraint. The wealth of the investor must be non-negative even after a jump. The duality method is not another alternative to solve this problem. In fact the primal problem leads to a HJB equation with boundary conditions. Because of the state space constraints these boundary conditions are not obvious to obtain. However, these delicate boundary conditions disappear in the dual problem. The regularity of the dual value function is not obvious to obtain. This explains the use of the notion of discontinuous viscosity solutions. The comparison theorem is not proved in a general framework since the operator which appears in the HJBVI contains an inf on a unbounded set which makes it discontinuous. In this paper, we prove the comparison theorem only when the space of claims is a finite one.

The paper is organized as follows. Section 2 describes the model. In Section 3, we formulate the dual optimization problem and we derive the associated HJBVI for the value function. In Section 4, we prove that the dual value function is a viscosity solution of our HJBVI. In Section 5, we prove a comparison theorem.

2 Problem formulation

Let (Ω, \mathcal{F}, P) be a complete probability space. We assume that the claims are generated by a compound Poisson process. More precisely, we consider an integer-valued random measure $\mu(dt, dz)$ with compensator $\pi(dz)dt$. We assume that $\pi(dz) = \varrho G(dz)$ where $G(dz)$ is a probability distribution on the bounded set $C \subseteq \mathbb{R}_+$ and ϱ is a positive constant. In this

case, the integral, with respect to the random measure $\mu(dt, dz)$, is simply a compound Poisson process: we have $\int_0^t \int_C z \mu(du, dz) = \sum_{i=1}^{N_t} Z_i$, where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity ϱ and $\{Z_i, i \in \mathbb{N}\}$ is a sequence of random variables with common distribution G which represent the claim sizes.

Let $T > 0$ be a finite time horizon. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ the filtration generated by the random measure $\mu(dt, dz)$.

By definition of the intensity $\pi(dz)dt$, the compensated jump process:

$$\tilde{\mu}(dt, dz) := \mu(dt, dz) - \pi(dz)dt$$

is such that $\{\tilde{\mu}([0, t] \times B), 0 \leq t \leq T\}$ is a (P, \mathbb{F}) martingale for all $B \in \mathcal{C}$, where \mathcal{C} is the Borel σ -field on C .

An insurance strategy is a predictable process $\theta = (\theta_t)_{0 \leq t \leq T}$ which represents the rate of insurance covered by the agent. We assume that the insurance premium is an affine function of the insurance strategy. Given an initial wealth $x \geq 0$ at time t and an insurance strategy θ , the wealth process of the agent at time $s \in [t, T]$ is then given by :

$$X_s^{t,x,\theta} := x + \int_t^s (\alpha - \beta(1 - \theta_u)) du - \int_t^s \int_C \theta_u z \mu(du, dz). \quad (2.1)$$

We assume that $\alpha \geq \beta \geq 0$ which means that the premium rate received by the agent is lower than the premium rate paid to the insurer. In the literature, this problem is known as a proportional reinsurance one. The agent is an insurer who has to pay a premium to the reinsurer. We impose that the insurance strategy satisfies:

$$\theta_s \in [0, 1] \quad \text{a.s. for all } t \leq s \leq T. \quad (2.2)$$

We also impose the following non-bankruptcy constraint on the wealth process:

$$X_s^{t,x,\theta} \geq 0 \quad \text{a.s. for all } t \leq s \leq T. \quad (2.3)$$

Given an initial wealth $x \geq 0$ at time t , an admissible policy θ is a predictable stochastic process $(\theta_s)_{t \leq s \leq T}$, such that conditions (2.2) and (2.3) are satisfied. We denote by $\mathcal{A}(t, x)$ the set of all admissible policies and $\mathcal{S}(t, x) := \{X^{t,x,\theta} \text{ such that } \theta \in \mathcal{A}(t, x)\}$.

Our agent has preferences modeled by a utility function U .

We assume that the agent's utility is described by a CRRA utility function i.e. $U(x) = \frac{x^\eta}{\eta}$, where $\eta \in (0, 1)$.

We denote by I the inverse of U' and we introduce the conjugate function of U defined by

$$\begin{aligned} \tilde{U}(y) &:= \sup_{x>0} \{U(x) - xy\}, \quad y > 0 \\ &= U(I(y)) - yI(y). \end{aligned} \quad (2.4)$$

A straightforward calculus shows that $\tilde{U}(y) = \frac{y^{-\gamma}}{\gamma}$ where $\gamma = \frac{\eta}{1-\eta}$ and $\tilde{U}'(y) = -I(y)$ for all $y > 0$.

The objective of the agent is to find the value function which is defined as

$$v(t, x) := \sup_{\theta \in \mathcal{A}(t, x)} E(U(X_T^{t, x, \theta})). \quad (2.5)$$

Notations: The constants which appear in the paper are generic and could change from line to line.

3 Dual optimization problem

First we introduce some notations. Let $x \geq 0$ and $t \in [0, T]$. We denote by $\mathcal{P}(\mathcal{S}(t, x))$ the set of all probability measures $Q \sim P$ with the following property: there exists $A \in \mathcal{I}_p$, set of non-decreasing predictable processes with $A_0 = 0$, such that :

$$X - A \text{ is a } Q - \text{local super-martingale for any } X \in \mathcal{S}(t, x). \quad (3.1)$$

The upper variation process of $\mathcal{S}(t, x)$ under $Q \in \mathcal{P}(\mathcal{S}(t, x))$ is the element $\tilde{A}^{\mathcal{S}(t, x)}(Q)$ in \mathcal{I}_p satisfying (3.1) and such that $A - \tilde{A}^{\mathcal{S}(t, x)}(Q) \in \mathcal{I}_p$ for any $A \in \mathcal{I}_p$ satisfying (3.1).

From Lemma 2.1 of Föllmer and Kramkov [13], we can derive $\mathcal{P}(\mathcal{S}(t, x))$ and $\tilde{A}^{\mathcal{S}(t, x)}(Q)$. This result states that $Q \in \mathcal{P}(\mathcal{S}(t, x))$ iff there is an upper bound for all the predictable processes arising in the Doob-Meyer decomposition of the special semi-martingale $V \in \mathcal{S}(t, x)$ under Q . In this case, the upper variation process is equal to this upper bound.

It is well-known from the martingale representation theorem for random measures (see e.g. Brémaud [2]) that all probability measures $Q \sim P$ have a density process in the form :

$$Z_s^\rho = \mathcal{E} \left(\int_t^s \int_C (\rho_u(z) - 1) \tilde{\mu}(du, dz) \right), \quad s \in [t, T], \quad (3.2)$$

where $\rho \in \mathcal{U}_t = \{(\rho_s(z))_{t \leq s \leq T} \text{ predictable process} : \rho_s(z) > 0, \text{ a.s., } t \leq s \leq T, z \in C, \int_t^T \int_C (|\log \rho_s(z)| + \rho_s(z) \pi(dz)) ds < \infty \text{ and } E[Z_T^\rho] = 1\}$.

By Girsanov's theorem, the predictable compensator of an element $X^\theta \in \mathcal{S}(t, x)$ under $P^\rho = Z_T^\rho \cdot P$ is :

$$A_s^{\rho, \theta} = \int_t^s (\alpha - \beta) du + \int_t^s \theta_u (\beta - \int_C \rho_u(z) z \pi(dz)) du.$$

We deduce from Lemma 2.1 of Föllmer and Kramkov [13] that $\mathcal{P}(\mathcal{S}(t, x)) = \{P^\rho : \rho \in \mathcal{U}_t\}$ and the upper variation process of P^ρ is :

$$\tilde{A}_s^{\mathcal{S}(t, x)}(P^\rho) = \int_t^s (\alpha - \beta) du + \int_t^s (\beta - \int_C \rho_u(z) z \pi(dz))_+ du.$$

From the non-decreasing property of U , we have

$$v(t, x) = \sup_{H \in \mathcal{C}_+(t, x)} E[U(H)],$$

where $\mathcal{C}_+(t, x) = \{H \in L_+^0(\mathcal{F}_T) : X_T^{t,x,\theta} \geq H \text{ a.s. for } \theta \in \mathcal{A}(t, x)\}$. It is easy to check the convexity property of the family $\mathcal{C}_+(t, x)$. For the closure property of this family in the semi-martingale topology, we refer to Pham [20].

The semi-martingale topology is associated to the Emery distance between two semi-martingales \tilde{X}^1 and \tilde{X}^2 defined as :

$$D_E(\tilde{X}^1, \tilde{X}^2) := \sum_{n \geq 1} 2^{-n} E \left[\sup_{0 \leq t \leq T \wedge n} |\tilde{X}_t^1 - \tilde{X}_t^2| \wedge 1 \right].$$

We refer to Mémin [17] for details on the semi-martingale topology. Since $\mathcal{C}_+(t, x)$ is convex and closed and using the optional decomposition under constraints of Föllmer and Kramkov [13], Mnif and Pham [19] gave the following dual characterization of the set $\mathcal{C}_+(t, x)$

$$\begin{aligned} H \in \mathcal{C}_+(t, x) \\ \iff J(H) := \sup_{Z \in \mathcal{P}^0(t, x), \tau \in \mathcal{T}_t} E \left[Z_T H 1_{\tau=T} - \int_t^\tau Z_u (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \right] \leq x, \end{aligned} \quad (3.3)$$

where $\mathcal{P}^0(t, x)$ is the subset of elements $P^\rho \in \mathcal{P}(\mathcal{S}(t, x))$ such that $\tilde{A}_T^{\mathcal{S}(t, x)}(P^\rho)$ is bounded and \mathcal{T}_t is the set of all stopping times valued in $[0, T]$.

As a corollary (see their corollary 4.1), they deduce that the set of admissible insurance strategies $\mathcal{A}(0, x)$ is non empty iff $x \geq b := (\beta - \alpha)T$.

Remark 3.1 *If the agent initial wealth is equal to $x = b$ and since $\mathcal{A}(0, x)$ is not empty, then the wealth process is given by $X_t^{0,x,\theta} = (\beta - \alpha)(T - t)$ for all $0 \leq t \leq T$ and so the only admissible strategy is $\theta_t = 0$ for all $0 \leq t \leq T$ which implies that the dynamic version of the value function satisfies $v(t, (\beta - \alpha)(T - t)) = 0$ for all $0 \leq t \leq T$. These boundary conditions obtained from the duality approach are not obvious from the primal approach.*

Now, we fix some initial wealth $x \geq b$. We make the following assumption

Assumption 3.1 *We assume that there exist $\bar{\gamma} \in (0, 1)$, $\bar{Q} \in \mathcal{P}^0(t, x)$ with density $\bar{Z}_T = \frac{d\bar{Q}}{dP}$ satisfying*

$$(\bar{Z}_T)^{-1} \in L^{\bar{p}}(P)$$

for some $\bar{p} > \frac{\bar{\gamma}}{1-\bar{\gamma}}$.

Under Assumption 3.1, Existence and uniqueness of problem (2.5) are proved (see Mnif and Pham [19]). We focus now on the study of the dual formulation.

The two following lemmas allow us to give an expression of the dual value function. We denote by \mathcal{D}_t the set of nonnegative, nonincreasing predictable and càdlàg processes $D = (D_s)_{t \leq s \leq T}$ with $D_t = 1$,

$$\mathcal{Y}^0(t) := \{Y^{\rho, D} = Z^\rho D, Z^\rho \in \mathcal{P}^0(t, x), D \in \mathcal{D}_t\}$$

for all $t \in [0, T]$ and $L_+^0(\mathcal{F}_T)$ is the set of nonnegative \mathcal{F}_T -measurable random variables.

Remark 3.2 We omit the dependence of $\mathcal{Y}^0(t)$ in the initial wealth x , since x is fixed in all the paper.

Remark 3.3 The set \mathcal{D}_t is introduced in the paper of Elkaroui and Jeanblanc [8] in the continuous case. In our case, we must enlarge this set to include càdlàg processes and extend some results of Mnif and Pham [19].

Lemma 3.1 *For all $x \geq b$, $X \in \mathcal{S}(t, x)$, $Y = ZD$, $Z \in \mathcal{P}^0(t, x)$, $D \in \mathcal{D}_t$, the processes :*

$$Z.X. - \int_t^\cdot Z_u(\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du,$$

$$\text{and } Y.X. - \int_t^\cdot Y_u(\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du$$

are supermartingales under P .

Proof. By definition of $\mathcal{P}^0(t, x)$, the process $Z.(X. - \int_t^\cdot (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du)$ is a P -local supermartingale. From Theorem VII.35 in Dellacherie and Meyer [7], the process

$$M = Z.X. - \int_t^\cdot Z_u(\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \quad (3.4)$$

is a P -local supermartingale. Moreover, M is bounded from below by the random variable $-\int_t^\cdot Z_u(\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du$, which is integrable under P . We deduce by Fatou's lemma that M is a P -supermartingale. On the other hand, by Itô's product rule and since D is predictable with finite variation, we get :

$$\begin{aligned} & Y_T X_T - \int_t^T Y_u(\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \\ = & Y_t X_t + \int_t^T D_u - dM_u + \int_t^T Z_u - X_u - dD_u + \sum_{t \leq u \leq T} \Delta M_u \Delta D_u. \end{aligned} \quad (3.5)$$

From Equation (3.4), we have

$$\Delta M_s \neq 0 \text{ iff } s = T_i = \inf\{u \geq t \text{ s.t. } N_u = i\}, \quad i \in \mathbb{N}$$

where T_i is the time of the claim number i and N is the Poisson process representing the number of claims. If the stopping time T_i is accessible, then from Theorem 15.1 in Rogers and Williams [21], we have $Z_{T_i^-} = Z_{T_i}$ which is false and so $(T_i)_{i \in \mathbb{N}}$ is a sequence of totally inaccessible stopping times. On the other hand, D is predictable process and so from Lemma 27.3 in Rogers and Williams [21] we have $\Delta D_\tau = 0$ for every totally inaccessible stopping time τ . Since $\Delta M_s = 0$ if $s \neq T_i$ and $\Delta D_s = 0$ if s is a totally inaccessible stopping time, we have

$$\Delta M_s \Delta D_s = 0 \text{ } ds \otimes dP \text{ a.s.} \quad (3.6)$$

Since D is nonnegative and nonincreasing, and Z, X are nonnegative, this shows that the process $: Y.X. - \int_t^\cdot Y_u(\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du$ is a P -local supermartingale, bounded from below by an $L^1(P)$ random variable, and hence a P -supermartingale. \square

Lemma 3.2 For all $H \in L_+^0(\mathcal{F}_T)$, we have :

$$J(H) = \sup_{Y \in \mathcal{Y}^0(t)} E \left[Y_T H - \int_t^T Y_u (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \right] \leq x \quad (3.7)$$

Proof. Fix some $H \in L_+^0(\mathcal{F}_T)$. Given an arbitrary $\tau \in \mathcal{T}_t$, we define a sequence $(D^n)_n$ of elements in \mathcal{D}_t by :

$$D_s^n = \exp \left(- \int_t^s n 1_{\tau \leq u} du \right), \quad t \leq s \leq T, \quad n \in \mathbb{N}.$$

We then have for all $Z \in \mathcal{P}^0(t, x)$:

$$\begin{aligned} & E \left[Z_T D_T^n H - \int_t^T Z_u D_u^n (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \right] \\ & \leq \sup_{Z \in \mathcal{P}^0(t, x), D \in \mathcal{D}_t} E \left[Z_T D_T H - \int_0^T Z_u D_u (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \right]. \end{aligned}$$

Since $D_u^n \rightarrow 1_{u \leq \tau}$ a.s., for all $t \leq u \leq T$, we have by Fatou's lemma :

$$\begin{aligned} & E \left[Z_T H 1_{\tau=T} - \int_0^\tau Z_u (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \right] \\ & \leq \sup_{Z \in \mathcal{P}^0(t, x), D \in \mathcal{D}_t} E \left[Z_T D_T H - \int_t^T Z_u D_u (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \right]. \end{aligned}$$

Identifying a probability measure $Q \in \mathcal{P}^0(t, x)$ with its density process Z , we then obtain from Bayes formula :

$$\begin{aligned} J(H) &= \sup_{Q \in \mathcal{P}^0(t, x), \tau \in \mathcal{T}_t} E^Q \left[H 1_{\tau=T} - \int_0^\tau (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \right] \\ &\leq \sup_{Z \in \mathcal{P}^0(t, x), D \in \mathcal{D}_t} E \left[Z_T D_T H - \int_0^T Z_u D_u (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \right]. \end{aligned}$$

Conversely, by the supermartingale property of $Y.X. - \int_0^\cdot Y_u (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du$ for any $Y \in \mathcal{Y}^0(t)$, see Lemma 3.1, we have :

$$\sup_{Y \in \mathcal{Y}^0(t)} E \left[Z_T D_T H - \int_t^T Z_u D_u (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \right] \leq x, \quad \forall H \in \mathcal{C}_+(x) \quad (3.8)$$

Now, by characterization (3.3), any $H \in L_+^0(\mathcal{F}_T)$ lies in $\mathcal{C}_+(J(H))$. We then deduce from inequality (3.8) that

$$\sup_{Y \in \mathcal{Y}^0(t)} E \left[Y_T H - \int_0^T Y_u (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \right] \leq J(H),$$

which proves the required equality (3.7). \square

The dual problem of (2.5) is written as:

$$\tilde{v}(t, y) := \inf_{Y \in \mathcal{Y}^0(t)} E \left[\tilde{U}(y Y_T^{\rho, D}) + \int_t^T y Y_u^{\rho, D} (\alpha - \beta + (\beta - \int_C \rho_u(z) z \pi(dz))_+) du \right], \quad (3.9)$$

We shall adopt a dynamic programming principle approach to study the dual value function (3.9). We recall the dynamic programming principle for our stochastic control problem: for any stopping time $0 \leq \tau \leq T$, $0 \leq t \leq T$ and $0 \leq h \leq T - t$,

$$\begin{aligned} \tilde{v}(t, y) &= \inf_{Y^{\rho, D} \in \mathcal{Y}^0(t)} E \left[\tilde{v} \left((t+h) \wedge \tau, Y_{(t+h) \wedge \tau}^{\rho, D} \right) \right. \\ &\quad \left. + \int_t^{(t+h) \wedge \tau} Y_u^{\rho, D} \left(\alpha - \beta + \left(\beta - \int_C \rho_u(z) z \pi(dz) \right)_+ \right) du \right], \end{aligned} \quad (3.10)$$

where $a \wedge b = \min(a, b)$ (see e.g. Fleming and Soner [12]).

Lemma 3.3 *Let $t \in [0, T]$ and $Y^{\rho, D} \in \mathcal{Y}^0(t)$. Then the process $Y^{\rho, D}$ evolves according to the following stochastic differential equation*

$$dY_s^{\rho, D} = Y_{s-}^{\rho, D} \left(-dL_s + \int_C (\rho_s(z) - 1) \tilde{\mu}(ds, dz) \right), \quad (3.11)$$

with

$$dL_s = -\frac{dD_s}{D_s} 1_{\{D_s > 0\}}, \quad t \leq s \leq T, \quad L_{t-} = 0. \quad (3.12)$$

Proof. By Itô's product rule, we have

$$dY_s^{\rho, D} = Z_{s-} dD_s + D_{s-} dZ_s + \triangle Z_s \triangle D_s.$$

From Equation (3.2), we have

$$\triangle Z_s \neq 0 \text{ iff } s = T_i = \inf\{u \geq t \text{ s.t. } N_u = i\}, \quad i \in \mathbb{N}.$$

Repeating the same argument as in equation (3.4), we have $\triangle Z_s \triangle D_s = 0 \, ds \otimes dP$ a.s. and so equation (3.11) is proved. \square

We denote by \mathcal{L}_t the set of adapted processes $(L_s)_{t \leq s \leq T}$ with possible jump at time $s = t$ and satisfying equation (3.12).

The Hamilton Jacobi Bellman Variational Inequality arising from the dynamic programming principle (3.10) is written as

$$\min \left\{ \frac{\partial \tilde{v}}{\partial t}(t, y) + H \left(t, y, \tilde{v}, \frac{\partial \tilde{v}}{\partial y} \right), -\frac{\partial \tilde{v}}{\partial y}(t, y) \right\} = 0, \quad (t, y) \in [0, T) \times (0, \infty), \quad (3.13)$$

with terminal condition

$$\tilde{v}(T, y) = \tilde{U}(y), \quad y \in (0, \infty), \quad (3.14)$$

where

$$H \left(t, y, \tilde{v}, \frac{\partial \tilde{v}}{\partial y} \right) := \inf_{\rho \in \Sigma} \left\{ A^\rho \left(t, y, \tilde{v}, \frac{\partial \tilde{v}}{\partial y} \right) + y \left(\alpha - \beta + \left(\beta - \int_C \rho(z) z \pi(dz) \right)_+ \right) \right\},$$

$$A^\rho \left(t, y, \tilde{v}, \frac{\partial \tilde{v}}{\partial y} \right) := \int_C \left(\tilde{v}(t, \rho(z)y) - \tilde{v}(t, y) - (\rho(z) - 1)y \frac{\partial \tilde{v}}{\partial y}(t, y) \right) \pi(dz),$$

and $\Sigma := \left\{ \rho \text{ positive Borel function defined on } C \text{ s.t. } \int_C (|\log \rho(z)| + \rho(z)) \pi(dz) < \infty \right\}$. This divides the time-space solvency region $[0, T] \times (0, \infty)$ into a no-jump region

$$R_1 = \left\{ (t, y) \in [0, T] \times (0, \infty), \text{ s.t. } \frac{\partial \tilde{v}}{\partial t}(t, y) + H \left(t, y, \tilde{v}, \frac{\partial \tilde{v}}{\partial y} \right) = 0 \right\}$$

and a jump region

$$R_2 = \left\{ (t, y) \in [0, T] \times (0, \infty), \text{ s.t. } \frac{\partial \tilde{v}}{\partial y}(t, y) = 0 \right\}.$$

4 Viscosity solution

In this section, we provide a rigorous characterization of the dual value function \tilde{v} as a viscosity solution to the HJBVI (3.13). The function \tilde{v} is not known to be continuous and so we shall work with the notion of discontinuous viscosity solutions. We prove that the dual value function lies in the set of functions $D_\gamma([0, T] \times (0, \infty))$ defined as follows:

$$D_\gamma([0, T] \times (0, \infty)) := \left\{ f : [0, T] \times (0, \infty) \rightarrow \mathbb{R} \text{ such that } , \right. \\ \left. \sup_{y>0} \frac{|f(t, y)|}{y + y^{-\gamma}} < \infty \text{ and } \sup_{x>0, y>0} \frac{|f(t, x) - f(t, y)|}{|x - y|(1 + x^{-(\gamma+1)} + y^{-(\gamma+1)})} < \infty \right\}.$$

The following lemma gives some properties of the dual value function \tilde{v} .

Lemma 4.1 *We assume that there exists a solution the the dual problem (3.9). The following properties hold:*

- 1) *The dual value function \tilde{v} is convex in y ,*
- 2) *The dual value function \tilde{v} satisfies the following growth condition*

$$\sup_{y>0} \frac{|\tilde{v}(t, y)|}{y + y^{-\gamma}} < \infty \tag{4.1}$$

- 3) *The dual value function \tilde{v} satisfies*

$$\sup_{y_1>0, y_2>0} \frac{|\tilde{v}(t, y_1) - \tilde{v}(t, y_2)|}{|y_1 - y_2|(1 + y_1^{-(\gamma+1)} + y_2^{-(\gamma+1)})} < \infty. \tag{4.2}$$

Proof. See Appendix. □

Remark 4.1 *When we assume that there exists a solution the the dual problem (3.9), one can use the conjugate duality relation proved in theorem 5.1 of Mnif and Pham [19].*

Since the dual value function \tilde{v} is locally bounded, the upper and the lower semi-continuous envelope of the function \tilde{v} are well-defined. The definition of the upper and the lower semi-continuous envelope of a function ϕ are given as follows.

Definition 4.1 (i) *The upper semi-continuous envelope of a function ϕ is*

$$\phi^*(t, y) = \limsup_{\substack{(t', y') \rightarrow (t, y) \\ t' \in [0, T], y' > 0}} \phi(t', y'), \text{ for all } (t, y) \in [0, T) \times (0, \infty). \quad (4.3)$$

(ii) *The lower semi-continuous envelope of a function ϕ is*

$$\phi_*(t, y) = \liminf_{\substack{(t', y') \rightarrow (t, y) \\ t' \in [0, T], y' > 0}} \phi(t', y'), \text{ for all } (t, y) \in [0, T) \times (0, \infty). \quad (4.4)$$

Since the continuity of the Hamiltonian H in his arguments is not obvious, we define lower semi-continuous envelope of H by

$$H_*(t, y, \psi, \frac{\partial \psi}{\partial y}) := \liminf_{\substack{(t', y') \rightarrow (t, y) \\ t' \in [0, T], y' > 0}} H(t', y', \psi, \frac{\partial \psi}{\partial y}) \text{ for all } (t, y) \in [0, T) \times (0, \infty).$$

Adapting the notion of viscosity solutions introduced by Crandall and Lions [4] and then by Soner [23] for first integrodifferential operators, we define the viscosity solution as follows:

Definition 4.2 (i) *A function ϕ is a viscosity supersolution of (3.13) in $[0, T) \times (0, \infty)$ if*

$$\min \left\{ \frac{\partial \psi}{\partial t}(\bar{t}, \bar{y}) + H_* \left(\bar{t}, \bar{y}, \psi, \frac{\partial \psi}{\partial y} \right), -\frac{\partial \psi}{\partial y}(\bar{t}, \bar{y}) \right\} \leq 0, \quad (4.5)$$

whenever $\psi \in C^1([0, T] \times (0, \infty))$ and $\phi_ - \psi$ has a strict global minimum at $(\bar{t}, \bar{y}) \in [0, T) \times (0, \infty)$.*

(ii) *A function ϕ is a viscosity subsolution of (3.13) in $[0, T) \times (0, \infty)$ if*

$$\min \left\{ \frac{\partial \psi}{\partial t}(\bar{t}, \bar{y}) + H \left(\bar{t}, \bar{y}, \psi, \frac{\partial \psi}{\partial y} \right), -\frac{\partial \psi}{\partial y}(\bar{t}, \bar{y}) \right\} \geq 0, \quad (4.6)$$

whenever $\psi \in C^1([0, T] \times (0, \infty))$ and $\phi^ - \psi$ has a strict global maximum at $(\bar{t}, \bar{y}) \in [0, T) \times (0, \infty)$.*

(iii) *A function ϕ is a viscosity solution of (3.13) in $[0, T) \times (0, \infty)$ if it is both super-solution and sub-solution in $[0, T) \times (0, \infty)$.*

Remark 4.2 *As it is seen in the definition of viscosity solutions, we use the lower semi-continuous envelope of the Hamiltonian H . In fact to prove that the dual value function \tilde{v} is viscosity solution of our HJBVI, we need only the regularity of H_* to derive inequality (6.24).*

The following theorem relates the dual value function \tilde{v} to the HJBVI (3.13).

Theorem 4.1 *The dual value function \tilde{v} is a viscosity solution of (3.13) in $[0, T) \times (0, \infty)$.*

Proof. See Appendix. □

The HJBVI (3.13) associated to our problem does not provide a complete characterization of the dual value function \tilde{v} . We need to specify the terminal condition. From the definition of \tilde{v} , it's obvious that $\tilde{v}(T, y) = \tilde{U}(y)$. Since we use the notion of discontinuous viscosity solutions, we need to characterize $\tilde{v}^*(T, y)$ and $\tilde{v}_*(T, y)$ for all $y \in (0, \infty)$ which is the object of the following theorem.

Theorem 4.2 *The terminal conditions of the upper and lower semi-continuous envelope of \tilde{v} satisfy the following inequalities*

$$\tilde{v}^*(T, y) := \limsup_{\substack{(t', y') \rightarrow (T, y) \\ t' \in [0, T], y' > 0}} \tilde{v}(t', y') \leq \tilde{U}(y), \quad \text{for all } y \in (0, \infty), \quad (4.7)$$

$$\tilde{v}_*(T, y) := \limsup_{\substack{(t', y') \rightarrow (T, y) \\ t' \in [0, T], y' > 0}} \tilde{v}(t', y') \geq \tilde{U}(y), \quad \text{for all } y \in (0, \infty). \quad (4.8)$$

Proof. We first prove inequality (4.7). Suppose on the contrary that there exists a constant $\eta > 0$ such that $\tilde{v}^*(T, y) \geq \tilde{U}(y) + 2\eta$. From the definition of \tilde{v}^* , there exists a sequence $((t_n, y_n))_{n \in \mathbb{N}}$ such that $(t_n, y_n) \rightarrow (T, y)$ and $\tilde{v}(t_n, y_n) \rightarrow \tilde{v}^*(T, y)$ when n tends to infinity, which implies

$$E[\tilde{U}(y_n Y_T^{\rho_n, D_n}) + \int_{t_n}^T y_n Y_s^{\rho_n, D_n} (\alpha - \beta + (\beta - \int_C \rho_{n_s}(z) z \pi(dz))_+) ds] \geq \tilde{U}(y) + \eta,$$

for all $Y^{\rho_n, D_n} \in \mathcal{Y}^0(t_n)$. Choosing $\rho_{n_s} = 1$ and $D_{n_s} = 1$ for all $s \in [t_n, T]$, we obtain

$$\tilde{U}(y_n) + \int_{t_n}^T y_n (\alpha - \beta + (\beta - \int_C z \pi(dz))_+) ds \geq \tilde{U}(y) + \eta.$$

Sending n to infinity, we have $\tilde{U}(y) \geq \tilde{U}(y) + \eta$ which is wrong and so inequality (4.7) is proved.

We prove now inequality (4.8). From the definition of \tilde{v}_* , there exists a sequence $((t_n, y_n))_{n \in \mathbb{N}}$ such that $(t_n, y_n) \rightarrow (T, y)$ and $\tilde{v}(t_n, y_n) \rightarrow \tilde{v}_*(T, y)$ when n tends to infinity.

From the definition of the dynamic version of the value function, we have for all $x \geq (\beta - \alpha)(T - t_n)$

$$v(t_n, x) \geq U(x + (\alpha - \beta)(T - t_n)). \quad (4.9)$$

Let $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$U(x + (\alpha - \beta)(T - t_n)) \geq U(x) - \epsilon. \quad (4.10)$$

Using the conjugate duality relation of Theorem 5.1 of Mnif and Pham [19] and relations (4.9) and (4.10), we obtain

$$\begin{aligned} \tilde{v}(t_n, y_n) &= \max_{x \geq (\beta - \alpha)(T - t_n)} [v(t_n, x) - x y_n] \\ &\geq \max_{x \geq (\beta - \alpha)(T - t_n)} [U(x) - x y_n] - \epsilon. \end{aligned} \quad (4.11)$$

Since $(t_n, y_n) \rightarrow (T, y)$ when n tends to infinity, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, $I(y_n) \geq (\beta - \alpha)(T - t_n)$ and so $\max_{x \geq (\beta - \alpha)(T - t_n)} [U(x) - xy_n] = \tilde{U}(y_n)$. For $n \geq n_0 \vee n_1$, inequality (4.11) implies

$$\tilde{v}(t_n, y_n) \geq \tilde{U}(y_n) - \epsilon.$$

Sending n to infinity and ϵ to 0, we prove inequality (4.8). \square

5 Uniqueness

We turn now to uniqueness questions. Our next main result is a comparison principle for discontinuous viscosity solutions to the HJBVI (3.13). It states that we can compare a viscosity sub-solution and a viscosity super-solution to the HJBVI (3.13) on $[0, T) \times (0, \infty)$, provided that we can compare them at terminal date as usual in parabolic problems.

The main difficulty in the comparison theorem comes from the discontinuity of the Hamiltonian. Here, we assume that the claims take values in the set $C = \{\delta_1, \delta_2, \dots, \delta_d\}$, $\delta_i > 0$, $1 \leq i \leq d$. In this case the Hamiltonian contains an inf on a bounded set which makes it continuous. We denote by π_i , $1 \leq i \leq d$ the intensity of the Poisson process associated to the claim having the size δ_i . The set Σ is defined as follows :

$$\Sigma := \{\rho = (\rho_i)_{1 \leq i \leq d}, \rho_i > 0, 1 \leq i \leq d\}. \quad (5.1)$$

The Hamiltonian H is given by

$$H\left(t, y, \tilde{v}, \frac{\partial \tilde{v}}{\partial y}\right) = \inf_{\rho \in \Sigma} \left\{ A^\rho\left(t, y, \tilde{v}, \frac{\partial \tilde{v}}{\partial y}\right) + y \left(\alpha - \beta + \left(\beta - \sum_{i=1}^d \rho_i \delta_i \pi_i \right)_+ \right) \right\}, \quad (5.2)$$

where

$$A^\rho\left(t, y, \tilde{v}, \frac{\partial \tilde{v}}{\partial y}\right) := \sum_{i=1}^d \pi_i \left(\tilde{v}(t, \rho_i y) - \tilde{v}(t, y) - (\rho_i - 1)y \frac{\partial \tilde{v}}{\partial y}(t, y) \right).$$

The following lemma states some properties of the functions \tilde{v}_* and \tilde{v}^* .

Lemma 5.1 *We assume that there exists a solution to the dual problem (3.9). The following properties hold :*

- 1) *The functions \tilde{v}^* and \tilde{v}_* are convex in y .*
- 2) *The functions \tilde{v}^* and \tilde{v}_* are nonincreasing on $(0, \infty)$.*
- 3) *The functions \tilde{v}_* and \tilde{v}^* satisfy the following growth condition*

$$\sup_{y>0} \frac{|\tilde{v}_*(t, y)|}{y + y^{-\gamma}}, \sup_{y>0} \frac{|\tilde{v}^*(t, y)|}{y + y^{-\gamma}} < \infty \quad (5.3)$$

- 4) *The functions \tilde{v}_* and \tilde{v}^* satisfy*

$$\sup_{y_1>0, y_2>0} \frac{|\tilde{v}_*(t, y_1) - \tilde{v}_*(t, y_2)|}{|y_1 - y_2|(1 + y_1^{-(\gamma+1)} + y_2^{-(\gamma+1)})} < \infty.$$

$$\sup_{y_1 > 0, y_2 > 0} \frac{|\tilde{v}^*(t, y_1) - \tilde{v}^*(t, y_2)|}{|y_1 - y_2|(1 + y_1^{-(\gamma+1)} + y_2^{-(\gamma+1)})} < \infty.$$

Proof. 1) Fix $\lambda \in (0, 1)$ and $(y, y', y'') \in (0, \infty)^3$ such that $y = \lambda y' + (1 - \lambda)y''$. From the definition of $\tilde{v}^*(t, y)$, there exists a sequence $((t_n, y_n))_n$ such that $v(t_n, y_n) \rightarrow \tilde{v}^*(t, y)$ when n goes to infinity. We set $y_n = \lambda y' + (1 - \lambda)y''_n$, then we have $y''_n \rightarrow y''$ when n goes to infinity. Since \tilde{v} is convex in y , then we have

$$\tilde{v}(t_n, y_n) \leq \lambda \tilde{v}(t_n, y') + (1 - \lambda) \tilde{v}(t_n, y''_n) \quad (5.4)$$

Sending n to infinity, inequality (5.4) implies

$$\tilde{v}^*(t, y) \leq \lambda \tilde{v}^*(t, y') + (1 - \lambda) \tilde{v}^*(t, y''), \quad (5.5)$$

which is the desired result.

We turn to the convexity of \tilde{v}_* . From Theorem 5.1 of Mnif and Pham [19], we have

$$\tilde{v}(t, y) = \max_{x \geq (\beta - \alpha)(T - t)} [v(t, x) - xy]. \quad (5.6)$$

We fix $(t, y) \in [0, T] \times (0, \infty)$. From the definition of \tilde{v}_* , there exists a sequence $((t_n, y_n))_n$ such that $v(t_n, y_n) \rightarrow \tilde{v}_*(t, y)$ and $(t_n, y_n) \rightarrow (t, y)$ when n goes to infinity. From equation (5.6), we have $\tilde{v}(t_n, y_n) \geq v(t_n, x) - xy_n \geq v_*(t, x) - xy_n$ for all $x \geq (\beta - \alpha)(T - t_n)$. Sending n to ∞ , we have $\tilde{v}_*(t, y) \geq v_*(t, x) - xy$ for all $x \geq (\beta - \alpha)(T - t)$ and so

$$\tilde{v}_*(t, y) \geq \max_{x \geq (\beta - \alpha)(T - t)} [v_*(t, x) - xy]. \quad (5.7)$$

From Theorem 5.1 of Mnif and Pham [19], there exists \hat{x} such that

$$\tilde{v}(t, y) = v(t, \hat{x}) - \hat{x}y \quad (5.8)$$

and $\frac{\partial \tilde{v}}{\partial y}(t, y) = -\hat{x}$. We consider a sequence $((t_n, y_n))_n$ such that $v(t_n, y_n) \rightarrow \tilde{v}_*(t, y)$ and $(t_n, y_n) \rightarrow (t, y)$ when n goes to infinity. From equation (5.8), there exists \hat{x}_n such that $\tilde{v}(t_n, y_n) = v(t_n, \hat{x}_n) - \hat{x}_n y_n$. Since $\tilde{v} \in D_\gamma([0, T] \times (0, \infty))$, we have $\hat{x}_n := |\frac{\partial \tilde{v}}{\partial y}(t_n, y_n)| \leq K(1 + y_n^{-(\gamma+1)})$ where K is a positive constant independent of n . Since $y_n \rightarrow y$ when n goes to infinity, the sequence $(\hat{x}_n)_n$ is bounded, and so long a subsequence $\hat{x}_n \rightarrow \hat{x}$ when n goes to infinity and so we have

$$\lim_{n \rightarrow \infty} v(t_n, \hat{x}_n) = \tilde{v}_*(t, y) + \hat{x}y = v_*(t, \hat{x}) \quad (5.9)$$

From inequality (5.7) and equation (5.9), we deduce that

$$\tilde{v}_*(t, y) = \max_{x \geq (\beta - \alpha)(T - t)} [v_*(t, x) - xy].$$

and so \tilde{v}_* is convex in y .

2) Fix $(t, y) \in [0, T) \times (0, \infty)$. Since at time t , only $D \in \mathcal{D}_t$ could make a jump, we have $y = Y_t \geq Y_{t+}$. From the dynamic programming principle (3.10) we have

$$\tilde{v}(t, y) \leq \tilde{v}(t, y(1 - \delta)) \text{ for all } 0 < \delta < 1,$$

and so the dual value function \tilde{v} is non-increasing with respect to y . This yields that \tilde{v}_* and \tilde{v}^* are nonincreasing.

3) Fix $(t, y) \in [0, T) \times (0, \infty)$. From the definition of $\tilde{v}^*(t, y)$, there exists a sequence $((t_n, y_n))_n$ such that $v(t_n, y_n) \rightarrow \tilde{v}_*(t, y)$ when n goes to infinity. From the growth condition of \tilde{v} , we have

$$|\tilde{v}(t_n, y_n)| \leq K(y_n + y_n^{-\gamma}),$$

where K is a positive constant. Sending n to infinity, we obtain the desired result. We use similar arguments to prove that $\sup_{y>0} \frac{|\tilde{v}^*(t, y)|}{y + y^{-\gamma}} < \infty$.

4) We prove only the first inequality. The second one is obtained by using similar arguments. Fix $(t, y_1, y_2) \in [0, T) \times (0, \infty) \times (0, \infty)$. From the definition of $\tilde{v}^*(t, y_2)$, there exists a sequence $((t_n, y_{2,n}))_n$ such that $v(t_n, y_{2,n}) \rightarrow \tilde{v}_*(t, y_2)$ when n goes to infinity. From inequality (4.2), we have

$$\tilde{v}(t_n, y_1) - \tilde{v}(t_n, y_{2,n}) \leq K|y_1 - y_{2,n}|(1 + y_1^{-(\gamma+1)} + y_{2,n}^{-(\gamma+1)}),$$

where K is a positive constant. Since $\liminf_{n \rightarrow \infty} \tilde{v}(t_n, y_1) \geq \tilde{v}_*(t, y_1)$, we obtain after sending n to infinity

$$-\tilde{v}_*(t, y_2) \leq K|y_1 - y_2|(1 + y_1^{-(\gamma+1)} + y_2^{-(\gamma+1)}) - \tilde{v}_*(t, y_1).$$

Using similar arguments, we deduce the inverse inequality and so

$$\sup_{y_1>0, y_2>0} \frac{|\tilde{v}_*(t, y_1) - \tilde{v}_*(t, y_2)|}{|y_1 - y_2|(1 + y_1^{-(\gamma+1)} + y_2^{-(\gamma+1)})} < \infty.$$

□

Remark 5.1 One could prove the monotonicity of \tilde{v}^* by using viscosity solutions arguments. In fact for each $\epsilon > 0$, we define $W(t, y) = \tilde{v}(t, y) - \epsilon y$, $(t, y) \in [0, T) \times (0, \infty)$. The function W satisfies in the viscosity sense $\frac{\partial W}{\partial y} \leq -\epsilon$, i.e. for all $(y_0, \psi) \in ((0, \infty), C^1([0, T] \times (0, \infty)))$ such that

$$(W^* - \psi)(t, y_0) = \max_{y \in (0, \infty)} (W^* - \psi)(t, y) \quad (5.10)$$

and so we have $\frac{\partial \psi}{\partial y}(t, y_0) \leq -\epsilon$. This proves that $\psi(t, \cdot)$ is nonincreasing on a neighborhood $V(y_0)$. Let $(y_1, y_2) \in V(y_0)$, we want to prove that $W^*(t, y_1) \geq W^*(t, y_2)$.

Suppose that $W^*(t, y_1) < W^*(t, y_2)$. We consider the function $V(t, y) = W^*(t, y_1)$ which solves

$$\frac{\partial V}{\partial y} = 0 \text{ on } (y_1, y_2), \quad (5.11)$$

together with the boundary conditions $V(t, y_1) = V(t, y_2) = W^*(t, y_1)$. Since W is a viscosity subsolution of Equation (5.11), From the comparison theorem (see Barles [1], Theorem 2.7 in the case of continuous viscosity solutions), we have

$$\inf_{[y_1, y_2]} (V - W^*)(t, y) = \inf((V - W^*)(t, y_1), (V - W^*)(t, y_2)) = 0,$$

which implies $V(t, y) \geq W^*(t, y)$ for all $y \in [y_1, y_2]$. Since $W^*(t, y_0) \leq V(t, y_0) = W^*(t, y_1)$ and $\psi(t, y_0) > \psi(t, y_1)$, then

$$(W^* - \psi)(t, y_0) < (W^* - \psi)(t, y_1),$$

which contradicts (5.10). Sending $\epsilon \rightarrow 0^+$, we obtain the desired result.

Now, we are able to prove the following comparison principle :

Theorem 5.1 *Let \tilde{v}_1 (resp \tilde{v}_2) $\in D_\gamma([0, T] \times (0, \infty))$ be a viscosity subsolution (resp supersolution) of (3.13) in $[0, T] \times (0, \infty)$ such that $\tilde{v}_1^*(T, y) \leq \tilde{v}_{2*}(T, y)$. We assume that \tilde{v}_{2*} is convex and nonincreasing in y , then*

$$\tilde{v}_1^*(t, y) \leq \tilde{v}_{2*}(t, y), \text{ for all } (t, y) \in [0, T] \times (0, \infty) \quad (5.12)$$

Proof. See Appendix. □

By combining the previous results, we finally obtain the following PDE characterization of the dual value function.

Corollary 5.1 *We assume that there exists a solution the the dual problem (3.9). The dual value function \tilde{v} is the unique viscosity solution of (3.13) with terminal condition $\tilde{v}(T, y) = \tilde{U}(y)$ in the class of functions $D_\gamma([0, T] \times (0, \infty))$.*

Remark 5.2 *In this remark, we formally discuss the numerical implications of Corollary 5.1. We can solve numerically the associated HJBVI by using an algorithm based on policy iterations. Then thanks to a verification theorem, We characterize the optimal insurance strategy by the solution of the variational inequality. These results are the object of the paper Mnif[18].*

6 Appendix

6.1 Proof of Lemma 4.1

- 1) From Theorem 5.1 of Mnif and Pham [19], we have $\tilde{v}(t, y) = \max_{x \geq (\beta - \alpha)(T - t)} [v(t, x) - xy]$. The convexity property of \tilde{v} in y holds since it is the upper envelope of affine functions.
- 2) Since the controls $\rho_s = 1$ and $D_s = 1$, $s \in [t, T]$ lie in $\mathcal{U}_t \times \mathcal{D}_t$, we have

$$\tilde{v}(t, y) \leq \tilde{U}(y) + Ky, \quad (6.1)$$

where K is a constant.

Let $(Z^n := Z^{\rho^n}, D^n)$ be a minimizing sequence of $\tilde{v}(t, y)$. From the definition of these minimizing sequences, there exist ϵ_n and $n_0 \in \mathbb{N}$ such that $\epsilon_n \rightarrow 0$ when $n \rightarrow \infty$ and for all $n \geq n_0$, we have

$$\begin{aligned} \tilde{v}(t, y) &\geq E \left[\tilde{U}(y Z_T^n D_T^n) \right] \\ &+ y E \left[\int_t^T Z_u^n D_u^n (\alpha - \beta + (\beta - \int_C \rho_u^n(z) z \pi(dz))_+) du \right] - \epsilon_n. \end{aligned} \quad (6.2)$$

Since $\epsilon_n \rightarrow 0$ when $n \rightarrow \infty$, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we have $\epsilon_n \leq \tilde{U}(y) + y$. We recall That $\tilde{U}(y) \geq U(0^+) \geq 0$ and so $\tilde{U}(y) + y > 0$ since $y > 0$. Using the boundedness of D^n , Jensen's inequality and the martingale property of Z^n , we have:

$$\begin{aligned} E \left[\tilde{U}(y Z_T^n D_T^n) \right] &\geq \tilde{U}(y E[Z_T^n]) \\ &\geq \tilde{U}(y). \end{aligned} \quad (6.3)$$

For the second term of the r.h.s of inequality (6.2), since $D_s^n \leq 1$ for all $s \in [t, T]$, using Fubini's theorem and the martingale property of Z^n , we have

$$\begin{aligned} E \left[\int_t^T y Z_u^n D_u^n (\alpha - \beta + (\beta - \int_C \rho_u^n(z) z \pi(dz))_+) du \right] &\geq y(\alpha - \beta) E \left[\int_t^T Z_u^n D_u^n du \right] \\ &\geq y(\alpha - \beta) \int_t^T E[Z_u^n] du \\ &\geq K' y, \end{aligned} \quad (6.4)$$

where K' is a constant independent of y . Inequalities (6.3) and (6.4) imply that

$$\tilde{v}(t, y) \geq \tilde{U}(y) + K' y. \quad (6.5)$$

From inequalities (6.1) and (6.5), we deduce that

$$\sup_{y > 0} \frac{|\tilde{v}(t, y)|}{y + \frac{y^{-\gamma}}{\gamma}} < \infty \quad (6.6)$$

- 3) Using the convexity in y of the dual value function \tilde{v} , we have

$$\frac{|\tilde{v}(t, y_1) - \tilde{v}(t, y_2)|}{|y_1 - y_2|} \leq |\tilde{v}'_d(t, y_1)| + |\tilde{v}'_d(t, y_2)|, \quad (6.7)$$

where \tilde{v}'_d is the right-hand derivative with respect to the variable y . Let $(Z^n := Z^{\rho^n}, D^n)$ be a minimizing sequence of $\tilde{v}(t, y_1)$. Let $\delta > 0$ and $(Z'^n := Z'^{\rho^n}, D'^n)$ be a minimizing sequence of $\tilde{v}(t, y_1 + \delta)$. From the definition of these minimizing sequences, there exist ϵ_n , ϵ'_n and $n_0 \in \mathbb{N}$ such that $\epsilon_n \rightarrow 0$, $\epsilon'_n \rightarrow 0$ when $n \rightarrow \infty$ and for all $n \geq n_0$, we have

$$\begin{aligned} \tilde{v}(t, y_1) &\geq y_1^{-\gamma} E \left[\frac{(Z_T^n D_T^n)^{-\gamma}}{\gamma} \right] \\ &+ y_1 E \left[\int_t^T Z_u^n D_u^n \left(\alpha - \beta + \left(\beta - \int_C \rho_u^n(z) z \pi(dz) \right)_+ \right) du \right] - \epsilon_n, \end{aligned} \quad (6.8)$$

and

$$\begin{aligned} \tilde{v}(t, y_1 + \delta) &\geq (y_1 + \delta)^{-\gamma} E \left[\frac{(Z_T'^n D_T'^n)^{-\gamma}}{\gamma} \right] \\ &+ (y_1 + \delta) E \left[\int_t^T Z_u'^n D_u'^n \left(\alpha - \beta + \left(\beta - \int_C \rho_u'^n(z) z \pi(dz) \right)_+ \right) du \right] \\ &- \epsilon'_n. \end{aligned} \quad (6.9)$$

Using the definition of $\tilde{v}(t, y_1)$ and $\tilde{v}(t, y_1 + \delta)$, we have

$$\tilde{v}(t, y_1) \leq y_1^{-\gamma} E \left[\frac{(Z_T'^n D_T'^n)^{-\gamma}}{\gamma} \right] + y_1 E \left[\int_t^T Z_u'^n D_u'^n \left(\alpha - \beta + \left(\beta - \int_C \rho_u'^n(z) z \pi(dz) \right)_+ \right) du \right],$$

and

$$\begin{aligned} \tilde{v}(t, y_1 + \delta) &\leq (y_1 + \delta)^{-\gamma} E \left[\frac{(Z_T^n D_T^n)^{-\gamma}}{\gamma} \right] \\ &+ (y_1 + \delta) E \left[\int_t^T Z_u^n D_u^n \left(\alpha - \beta + \left(\beta - \int_C \rho_u^n(z) z \pi(dz) \right)_+ \right) du \right]. \end{aligned} \quad (6.10)$$

Using inequalities (6.8) and (6.10), we deduce that

$$\begin{aligned} &\frac{\tilde{v}(t, y_1 + \delta) - \tilde{v}(t, y_1)}{\delta} \\ &\leq \frac{(y_1 + \delta)^{-\gamma} - y_1^{-\gamma}}{\delta} E \left[\frac{(Z_T^n D_T^n)^{-\gamma}}{\gamma} \right] \\ &+ E \left[\int_t^T Z_u^n D_u^n \left(\alpha - \beta + \left(\beta - \int_C \rho_u^n(z) z \pi(dz) \right)_+ \right) du \right] + \frac{\epsilon_n}{\delta}. \end{aligned} \quad (6.11)$$

From inequality (6.3), we have

$$E \left[(Z_T^n D_T^n)^{-\gamma} \right] \geq K, \quad (6.12)$$

where K is a positive constant independent of y and δ . Since $\epsilon_n \rightarrow 0$ when $n \rightarrow \infty$, there exists n_1 such that for all $n \geq n_1$, we have $\frac{\epsilon_n}{\delta} \leq 1$. For the second term of the r.h.s of inequality (6.11), since $D_s \leq 1$ for all $s \in [t, T]$, using Fubini and martingale property of

Z , we have

$$\begin{aligned}
E \left[\int_t^T Z_u^n D_u^n \left(\alpha - \beta + \left(\beta - \int_C \rho_u^n(z) z \pi(dz) \right)_+ \right) du \right] &\leq \alpha E \left[\int_t^T Z_u^n D_u^n du \right] \\
&\leq \alpha \int_t^T E [Z_u^n] du \\
&\leq K.
\end{aligned} \tag{6.13}$$

Using inequalities (6.11), (6.12) and (6.13)

$$\frac{\tilde{v}(t, y_1 + \delta) - \tilde{v}(t, y_1)}{\delta} \leq K \left(\frac{(y_1 + \delta)^{-\gamma} - y_1^{-\gamma}}{\delta} + 1 \right).$$

Sending $\delta \rightarrow 0^+$, we have

$$\tilde{v}'_d(t, y_1) \leq K(-y_1^{-(\gamma+1)} + 1). \tag{6.14}$$

Similarly, we obtain that

$$\begin{aligned}
&\frac{\tilde{v}(t, y_1 + \delta) - \tilde{v}(t, y_1)}{\delta} \\
&\geq \frac{(y_1 + \delta)^{-\gamma} - y_1^{-\gamma}}{\delta} E \left[\frac{(Z_T^n D_T^n)^{-\gamma}}{\gamma} \right] \\
&+ E \left[\int_t^T Z_u^n D_u^n \left(\alpha - \beta + \left(\beta - \int_C \rho_u^n(z) z \pi(dz) \right)_+ \right) du \right] - \frac{\epsilon'_n}{\delta}.
\end{aligned} \tag{6.15}$$

We define $\tilde{\rho}_s := C^{\tilde{\rho}} = \frac{\beta}{\int_C z \pi(dz)}$ and $\tilde{D}_s = 1$ for all $s \in [t, T]$. From the definition of \tilde{v} and using the martingale property of $Z^{\tilde{\rho}}$, we have

$$\begin{aligned}
&\tilde{v}(t, y_1 + \delta) \\
&\leq (y_1 + \delta)^{-\gamma} E \left[\frac{(Z_T^{\tilde{\rho}} \tilde{D}_T)^{-\gamma}}{\gamma} \right] + (y_1 + \delta) E \left[\int_t^T Z_u^{\tilde{\rho}} \tilde{D}_u \left(\alpha - \beta + \left(\beta - \int_C \tilde{\rho}_u(z) z \pi(dz) \right)_+ \right) du \right] \\
&= (y_1 + \delta)^{-\gamma} E \left[\frac{(Z_T^{\tilde{\rho}})^{-\gamma}}{\gamma} \right] + (y_1 + \delta)(\alpha - \beta)(T - t).
\end{aligned} \tag{6.16}$$

From inequality (6.9), using the martingale of Z^n and since $0 \leq D_s^n \leq 1$ for all $s \in [t, T]$, we have

$$\tilde{v}(t, y_1 + \delta) \geq (y_1 + \delta)^{-\gamma} E \left[\frac{(Z_T^n D_T^n)^{-\gamma}}{\gamma} \right] + (y_1 + \delta)(\alpha - \beta)(T - t) - \epsilon'_n \tag{6.17}$$

from (6.16) and (6.17), we deduce that

$$E \left[(Z_T^n D_T^n)^{-\gamma} \right] \leq E \left[(Z_T^{\tilde{\rho}})^{-\gamma} \right] + \gamma \epsilon'_n (y_1 + \delta)^\gamma.$$

We know that $Z^{\tilde{\rho}}$ is given by the formula ¹

$$Z_T^{\tilde{\rho}} = \exp(\varrho(t - T)(C^{\tilde{\rho}} - 1))(C^{\tilde{\rho}})^{N_T - N_t}$$

¹ The solution of the SDE $dZ_t = Z_t dH_t$ is given by the Doléans Dade exponential formula $Z_t = \exp(H_t - \frac{1}{2} \langle H^c \rangle_t) \prod_{0 \leq s \leq t} (1 + \Delta H_s) \exp(-\Delta H_s)$ where $\langle H^c \rangle$ is the quadratic variation of the continuous part of H .

and so $E \left[(Z_T^{\tilde{\rho}})^{-\gamma} \right] < \infty$. Since $\epsilon'_n \rightarrow 0$ when $n \rightarrow \infty$, there exists n_2 such that for all $n \geq n_2$, we have $E [(Z_T^{\tilde{\rho}} D_T^{\tilde{\rho}})^{-\gamma}] \leq K'$, where K' is a positive constant independent of y_1 and δ . Using the boundedness of D , inequality (6.15) implies

$$\frac{\tilde{v}(t, y_1 + \delta) - \tilde{v}(t, y_1)}{\delta} \geq K' \left(\frac{(y_1 + \delta)^{-\gamma} - y_1^{-\gamma}}{\delta} - 1 \right).$$

Sending $\delta \rightarrow 0^+$, we have

$$\tilde{v}'_d(t, y_1) \geq -K'(y_1^{-(\gamma+1)} + 1). \quad (6.18)$$

Using (6.7), (6.14) and (6.18), we obtain

$$\sup_{y_1 > 0, y_2 > 0} \frac{|\tilde{v}(t, y_1) - \tilde{v}(t, y_2)|}{|y_1 - y_2| \left(1 + y_1^{-(\gamma+1)} + y_2^{-(\gamma+1)} \right)} < \infty$$

□

6.2 Proof of Theorem 4.1

We first prove that \tilde{v} is a viscosity sub-solution of (3.13) in $[0, T] \times (0, \infty)$.

Let $(t, y) \in [0, T] \times (0, \infty)$ and $\psi \in C^1([0, T] \times (0, \infty))$ such that without loss of generality

$$0 = (\tilde{v}^* - \psi)(t, y) = \max_{[0, T] \times (0, \infty)} (\tilde{v}^* - \psi).$$

From the definition of \tilde{v}^* , there exists a sequence $(t_n, y_n) \in [0, T] \times (0, \infty)$ such that $(t_n, y_n) \rightarrow (t, y)$ and $\tilde{v}(t_n, y_n) \rightarrow \tilde{v}^*(t, y)$ when $n \rightarrow \infty$.

For $\eta > 0$, $\rho_{n_s} = \tilde{\rho}$ a positive Borel function and $D_{n_s} = 1$ for all $s \geq t_n$, we set

$$\theta_n := \inf \{ t \geq t_n \text{ such that } (t, y_n Y_t^{\rho_n, D_n}) \notin B_\eta(t_n, y_n) \} \wedge T,$$

where $B_\eta(t_n, y_n) = \{(t, y) \in [0, T] \times (0, \infty) \text{ such that } |t - t_n| + |y - y_n| \leq \eta\}$. By the right continuity of the paths, we have $\theta_n > t_n$ a.s. For all $0 < h < T - t_n$, the dynamic programming principle

$$\begin{aligned} \tilde{v}(t_n, y_n) &= \inf_{Y^{\rho, D} \in \mathcal{Y}^0(t_n)} E \left[\tilde{v} \left((t_n + h) \wedge \theta_n, y_n Y_{(t_n + h) \wedge \theta_n}^{\rho, D} \right) \right. \\ &\quad \left. + \int_{t_n}^{(t_n + h) \wedge \theta_n} y_n Y_s^{\rho, D} \left(\alpha - \beta + \left(\beta - \int_C \rho_s(z) z \pi(dz) \right)_+ \right) ds \right] \end{aligned}$$

implies

$$\begin{aligned} \gamma_n + \psi(t_n, y_n) &\leq E \left[\psi \left((t_n + h) \wedge \theta_n, y_n Y_{(t_n + h) \wedge \theta_n}^{\rho_n, D_n} \right) \right. \\ &\quad \left. + \int_{t_n}^{(t_n + h) \wedge \theta_n} y_n Y_s^{\rho_n, D_n} \left(\alpha - \beta + \left(\beta - \int_C \rho_n(z) z \pi(dz) \right)_+ \right) ds \right], \end{aligned} \quad (6.19)$$

where the sequence $\gamma_n := \tilde{v}(t_n, y_n) - \psi(t_n, y_n)$ is determinist and converges to zero when n tends to infinity. Applying Itô's formula to $\psi(t_n + h, y_n Y_{t+h}^{\rho_n, D_n})$, we get

$$\begin{aligned} & E \left[\frac{1}{h} \int_{t_n}^{(t_n+h) \wedge \theta_n} \frac{\partial \psi}{\partial t} (s, y_n Y_s^{\rho_n, D_n}) + A^{\rho_n} \left(s, y_n Y_s^{\rho_n, D_n}, \psi, \frac{\partial \psi}{\partial y} \right) ds \right] \\ & + E \left[\frac{1}{h} \int_{t_n}^{(t_n+h) \wedge \theta_n} y_n Y_s^{\rho_n, D_n} \left(\alpha - \beta + \left(\beta - \int_C \rho_n(z) z \pi(dz) \right)_+ \right) ds \right] \\ & + E \left[\frac{1}{h} \int_{t_n}^{(t_n+h) \wedge \theta_n} \psi \left(s, \rho_n(z) y_n Y_s^{\rho_n, D_n} \right) - \psi \left(s, y_n Y_s^{\rho_n, D_n} \right) \tilde{\mu}(ds, dz) \right] \geq \frac{\gamma_n}{h}. \end{aligned}$$

By the martingale's property, the third expectation on the left hand-side of the last inequality vanishes and so we obtain

$$\begin{aligned} & E \left[\frac{1}{h} \int_{t_n}^{(t_n+h) \wedge \theta_n} \frac{\partial \psi}{\partial t} (s, y_n Y_s^{\rho_n, D_n}) + A^{\rho_n} \left(s, y_n Y_s^{\rho_n, D_n}, \psi, \frac{\partial \psi}{\partial y} \right) ds \right. \\ & \left. + \frac{1}{h} \int_{t_n}^{(t_n+h) \wedge \theta_n} y_n Y_s^{\rho_n, D_n} \left(\alpha - \beta + \left(\beta - \int_C \rho_n(z) z \pi(dz) \right)_+ \right) ds \right] \geq \frac{\gamma_n}{h}. \quad (6.20) \end{aligned}$$

From the definition of γ_n , two cases are possible:

★ Case 1: if the set $\{n \geq 0 : \gamma_n = 0\}$ is finite, then there exists a subsequence renamed $(\gamma_n)_{n \geq 0}$ such that $\gamma_n \neq 0$ for all n and we set $h = \sqrt{\gamma_n}$.

★ Case 2: if the set $\{n \geq 0 : \gamma_n = 0\}$ is not finite, then there exists a subsequence renamed $(\gamma_n)_{n \geq 0}$ such that $\gamma_n = 0$ for all n and we set $h = n^{-1}$.

In both cases $\frac{\gamma_n}{h} \rightarrow 0$ as n tends to ∞ . We now send n to infinity. The a.s. convergence of the random value inside the expectation is obtained by the mean value Theorem. Since $\int_C \pi(dz) < \infty$ and using the definition of θ_n , the random variable

$$\begin{aligned} & \frac{1}{h} \int_{t_n}^{(t_n+h) \wedge \theta_n} \frac{\partial \psi}{\partial t} (s, y_n Y_s^{\rho_n, D_n}) + A^{\rho_n} \left(s, y_n Y_s^{\rho_n, D_n}, \psi, \frac{\partial \psi}{\partial y} \right) \\ & + y_n Y_s^{\rho_n, D_n} \left(\alpha - \beta + \left(\beta - \int_C \rho_{ns}(z) z \pi(dz) \right)_+ \right) ds \end{aligned}$$

is essentially bounded, uniformly in n , on the stochastic interval $[t_n, (t_n + h) \wedge \theta_n]$. Sending n to infinity, it follows by the dominated convergence theorem

$$\frac{\partial \psi}{\partial t}(t, y) + A^{\tilde{\rho}}(t, y, \psi, \frac{\partial \psi}{\partial y}) + y \left(\alpha - \beta + \left(\beta - \int_C \tilde{\rho}(z) z \pi(dz) \right)_+ \right) \geq 0,$$

for all $\tilde{\rho} \in \Sigma$ and so

$$\frac{\partial \psi}{\partial t}(t, y) + H(t, y, \psi, \frac{\partial \psi}{\partial y}) \geq 0. \quad (6.21)$$

It remains to prove

$$-\frac{\partial \psi}{\partial y}(t, y) \geq 0.$$

For $\delta \in (0, 1)$, we set

$$L_s = \begin{cases} 0 & \text{if } s = t_n^- \\ \delta & \text{if } s \geq t_n, \end{cases}$$

and so the process D is a constant for all $s \geq t_n$. We choose $\rho_s(z) = \tilde{\rho}(z)$ for all $s \geq t_n$ and $z \in C$, where $\tilde{\rho} \in \Sigma$. We have $y_n Y^{\rho, D} \in \mathcal{Y}^0(t_n)$. Sending n to infinity, we have $y Y_{t^+}^{\rho, D} = y(1 - \delta)$. Sending $n \rightarrow \infty$ in (6.19), by the dominated convergence theorem we get

$$\psi(t, y) \leq \psi(t, y(1 - \delta)).$$

Sending $\delta \rightarrow 0^+$, we obtain

$$-\frac{\partial \psi}{\partial y}(t, y) \geq 0. \quad (6.22)$$

Combining (6.21) and (6.22), we conclude that \tilde{v} is a viscosity subsolution.

For supersolution inequality (4.5), let $\psi \in C^1([0, T] \times (0, \infty))$, $(\bar{t}, \bar{y}) \in [0, T] \times (0, \infty)$ such that $(\tilde{v}_* - \psi)(\bar{t}, \bar{y}) = \min(\tilde{v}_* - \psi) = 0$, we need to show

$$\min \left\{ \frac{\partial \psi(\bar{t}, \bar{y})}{\partial t} + H_* \left(\bar{t}, \bar{y}, \psi, \frac{\partial \psi}{\partial y} \right), -\frac{\partial \psi}{\partial y}(\bar{t}, \bar{y}) \right\} \leq 0. \quad (6.23)$$

Suppose the contrary. Hence the left-hand side of (6.23) is positive. By smoothness of ψ and since H_* is lower semi-continuous, there exist η and ϵ satisfying:

$$\min \left\{ \frac{\partial \psi(t, y)}{\partial t} + H_* \left(t, y, \psi, \frac{\partial \psi}{\partial y} \right), -\frac{\partial \psi}{\partial y}(t, y) \right\} \geq \epsilon, \quad (6.24)$$

for all $(t, y) \in B_\eta(\bar{t}, \bar{y})$, where $B_\eta(\bar{t}, \bar{y}) = \{(t, y) \in [0, T] \times (0, \infty) \text{ such that } |t - \bar{t}| + |y - \bar{y}| \leq \eta\}$. By changing η , we may assume that $B_\eta(\bar{t}, \bar{y}) \subset [0, T] \times (0, \infty)$.

Since (\bar{t}, \bar{y}) is a strict global minimizer of $\tilde{v}_* - \psi$, there exists $\xi > 0$ such that

$$\min_{(t, y) \in \partial B_\eta(\bar{t}, \bar{y})} (\tilde{v}_* - \psi)(t, y) = \xi,$$

which implies $\tilde{v}_*(t, y) \geq \xi + \psi(t, y)$ for all $(t, y) \in \partial B_\eta(\bar{t}, \bar{y})$ the parabolic boundary of $B_\eta(\bar{t}, \bar{y})$. From the definition of \tilde{v}_* , there exists a sequence $(t_n, y_n) \in [0, T] \times (0, \infty)$ such that $(t_n, y_n) \rightarrow (\bar{t}, \bar{y})$ and $\tilde{v}(t_n, y_n) \rightarrow \tilde{v}_*(\bar{t}, \bar{y})$ when $n \rightarrow \infty$. We suppose that $(t_n, y_n) \in B_\eta(\bar{t}, \bar{y})$. Let $Y^{\rho, D} \in \mathcal{Y}^0(t_n)$ be given and the stopping time θ_n defined by

$$\theta_n = \inf\{t \geq t_n \text{ such that } (t, y_n Y_t^{\rho, D}) \notin B_\eta(\bar{t}, \bar{y})\} \wedge T.$$

Since the control $L \in \mathcal{L}_{t_n}$ is singular with possible jump at $t = t_n$, the couple $(t, y_n Y_t^{\rho, D})$ might jump out of $B_\eta(\bar{t}, \bar{y})$ at t_n . If the control D makes alone the latter couple jump out of $B_\eta(\bar{t}, \bar{y})$, we set $\theta_n^D := \theta_n$ else $\theta_n^D := T$. In this case, the process Y decreases. We know, from the dynamic programming principle (3.10) that

$$\tilde{v}(t, y) \leq \tilde{v}(t, y(1 - \delta)) \text{ for all } 0 < \delta < 1,$$

and so the dual value function \tilde{v} is non-increasing with respect to y . However, the point Poisson process could contribute to the jump out of $B_\eta(\bar{t}, \bar{y})$. In this case the dual value function \tilde{v} is not necessarily non-increasing in the direction of the jump. The control ρ could also contribute to hit the boundary of $B_\eta(\bar{t}, \bar{y})$. To overcome this problem, we introduce θ_j the first time after t_n the state process Y jumps because of the point Poisson process and we set $\theta_n^\rho := \theta_n$ when $y_n Y^{\rho, D}$ jumps out $B_\eta(\bar{t}, \bar{y})$ because of the control ρ else $\theta_n^\rho := T$. We set also $\theta_p := \theta_n^\rho \wedge \theta_j$. Note that, by right continuity of the paths, we have $\theta_p > t_n$ a.s. Let θ be the stopping time defined as follows $\theta := \theta_n^D \wedge \theta_p$.

★ On the set $\{\theta_n^D < \theta_p\}$. Let (θ_n^D, y') be the intersection between $\partial B_\eta(\bar{t}, \bar{y})$ the parabolic boundary of $B_\eta(\bar{t}, \bar{y})$ and the line between $(\theta_n^D, y_n Y_{\theta_n^D}^{\rho, D})$ and $(\theta_n^D, y_n Y_{\theta_n^D}^{\rho, D})$. From (6.24), we deduce that ψ is non-increasing along this line in $\overline{B_\eta(\bar{t}, \bar{y})}$. Since the dual value function \tilde{v} is non-increasing with respect to y , we have

$$\tilde{v}(\theta_n^D, y_n Y_{\theta_n^D}^{\rho, D}) \geq \tilde{v}(\theta_n^D, y') \geq \psi(\theta_n^D, y') + \xi \geq \psi(\theta_n^D, y_n Y_{\theta_n^D}^{\rho, D}) + \xi$$

Using the inequality above and applying Itô's formula to $\psi(t, y_n Y_t^{\rho, D})$, we have

$$\begin{aligned} & \tilde{v}(\theta_n^D, y_n Y_{\theta_n^D}^{\rho, D}) \\ & \geq \psi(\theta_n^D, y_n Y_{\theta_n^D}^{\rho, D}) + \xi \\ & \geq \psi(t_n, y_n) + \int_{t_n}^{\theta_n^D} \frac{\partial \psi}{\partial t}(s, y_n Y_s^{\rho, D}) + A^\rho(s, y_n Y_s^{\rho, D}, \psi, \frac{\partial \psi}{\partial y}) ds - \int_{t_n}^{\theta_n^D} \frac{\partial \psi}{\partial y}(s, y_n Y_s^{\rho, D}) y_n Y_{s-}^{\rho, D} dL_s \\ & + \int_{t_n}^{\theta_n^D} \int_C \psi(s, \rho_s(z) y_n Y_{s-}^{\rho, D}) - \psi(s, y_n Y_{s-}^{\rho, D}) \tilde{\mu}(ds, dz) + \xi. \end{aligned} \tag{6.25}$$

For $t_n \leq s < \theta_n^D$, (6.24) implies:

$$\begin{aligned} & \frac{\partial \psi}{\partial t}(s, y_n Y_s^{\rho, D}) + A^\rho(s, y_n Y_s^{\rho, D}, \psi, \frac{\partial \psi}{\partial y}) \\ & + y_n Y_s^{\rho, D} \left(\alpha - \beta + (\beta - \int_C \rho_s(z) z \pi(dz))_+ \right) \geq 0, \end{aligned} \tag{6.26}$$

$$- \frac{\partial \psi}{\partial y}(s, y_n Y_s^{\rho, D}) \geq 0. \tag{6.27}$$

Substituting (6.26) and (6.27) into (6.25), we have

$$\begin{aligned} & \tilde{v}(\theta_n^D, y_n Y_{\theta_n^D}^{\rho, D}) + \int_{t_n}^{\theta_n^D} y_n Y_s^{\rho, D} \left(\alpha - \beta + (\beta - \int_C \rho_s(z) z \pi(dz))_+ \right) ds \\ & \geq \psi(t_n, y_n) + \int_{t_n}^{\theta_n^D} \int_C \psi(s, \rho_s(z) y_n Y_{s-}^{\rho, D}) - \psi(s, y_n Y_{s-}^{\rho, D}) \tilde{\mu}(ds, dz) + \xi. \end{aligned} \tag{6.28}$$

★ On the set $\{\theta_n^D \geq \theta_p\}$, we have

$$\begin{aligned}
\tilde{v}(\theta_p, y_n Y_{\theta_p}) &\geq \psi(\theta_p, y_n Y_{\theta_p}) \\
&\geq \psi(t_n, y_n) + \int_{t_n}^{\theta_p} \frac{\partial \psi}{\partial t}(s, y_n Y_s^{\rho, D}) + A^\rho(s, y_n Y_s^{\rho, D}, \psi, \frac{\partial \psi}{\partial y}) ds \\
&\quad - \int_{t_n}^{\theta_p} \frac{\partial \psi}{\partial y}(s, y_n Y_s^{\rho, D}) y_n Y_{s-}^{\rho, D} dL_s \\
&\quad + \int_{t_n}^{\theta_p} \int_C \psi(s, \rho_s(z) y_n Y_{s-}^{\rho, D}) - \psi(s, y_n Y_{s-}^{\rho, D}) \tilde{\mu}(ds, dz).
\end{aligned} \tag{6.29}$$

For $t_n \leq s < \theta_p$, (6.24) implies:

$$\begin{aligned}
\frac{\partial \psi}{\partial t}(s, y_n Y_s^{\rho, D}) &+ A^\rho(s, y_n Y_s^{\rho, D}, \psi, \frac{\partial \psi}{\partial y}) \\
&+ y_n Y_s^{\rho, D} \left(\alpha - \beta + (\beta - \int_C \rho_s(z) z \pi(dz))_+ \right) \geq \epsilon,
\end{aligned} \tag{6.30}$$

$$-\frac{\partial \psi}{\partial y}(s, y_n Y_s^{\rho, D}) \geq 0. \tag{6.31}$$

Substituting (6.30) and (6.31) into (6.29), we have

$$\begin{aligned}
&\tilde{v}(\theta_p, y_n Y_{\theta_p}) + \int_{t_n}^{\theta_p} y_n Y_s^{\rho, D} \left(\alpha - \beta + (\beta - \int_C \rho_s(z) z \pi(dz))_+ \right) ds \\
&\geq \psi(t_n, y_n) + \int_{t_n}^{\theta_p} \int_C \left(\psi(s, \rho_s(z) y_n Y_{s-}^{\rho, D}) - \psi(s, y_n Y_{s-}^{\rho, D}) \right) \tilde{\mu}(ds, dz) + \epsilon(\theta_p - t_n).
\end{aligned} \tag{6.32}$$

Putting the two cases (6.28) and (6.32) together, we get

$$\begin{aligned}
&E \left[\tilde{v}(\theta, y_n Y_{\theta}^{\rho, D}) + \int_{t_n}^{\theta} y_n Y_s^{\rho, D} \left(\alpha - \beta + (\beta - \int_C \rho_s(z) z \pi(dz))_+ \right) ds \right] \\
&\geq E \left[1_{\{\theta_n^D < \theta_p\}} \left(\tilde{v}(\theta_n^D, y_n Y_{\theta_n^D}^{\rho, D}) + \int_{t_n}^{\theta_n^D} y_n Y_s^{\rho, D} \left(\alpha - \beta + (\beta - \int_C \rho_s(z) z \pi(dz))_+ \right) ds \right) \right] \\
&+ E \left[1_{\{\theta_n^D \geq \theta_p\}} \left(\tilde{v}(\theta_p, y_n Y_{\theta_p}^{\rho, D}) + \int_{t_n}^{\theta_p} y_n Y_s^{\rho, D} \left(\alpha - \beta + (\beta - \int_C \rho_s(z) z \pi(dz))_+ \right) ds \right) \right] \\
&\geq \psi(t_n, y_n) + \xi P(\theta_n^D < \theta_p) + \epsilon E[1_{\{\theta_n^D \geq \theta_p\}}(\theta_p - t_n)]
\end{aligned} \tag{6.33}$$

Suppose that for all $\xi' > 0$, there exists $Y \in \mathcal{Y}^0(t_n)$ such that

$$\xi P(\theta_n^D < \theta_p) + \epsilon E[1_{\{\theta_n^D \geq \theta_p\}}(\theta_p - t_n)] \leq \xi' \tag{6.34}$$

Since $\theta_p > t_n$ a.s. and $0 \leq \epsilon E[1_{\{\theta_n^D \geq \theta_p\}}(\theta_p - t_n)] \leq \xi'$, for ξ' sufficiently small, we deduce that $\theta_n^D < \theta_p$ a.s. Inequality (6.34) implies $\xi \leq 0$ for ξ' sufficiently small which is false and so there exists $\zeta > 0$ such that for all $Y \in \mathcal{Y}^0(t_n)$, we have

$$\xi P(\theta_n^D < \theta_p) + \epsilon E[1_{\{\theta_n^D \geq \theta_p\}}(\theta_p - t_n)] \geq \zeta. \tag{6.35}$$

Inequalities (6.33) and (6.35) imply

$$\begin{aligned} & \tilde{v}(t_n, y_n) + \delta_n \\ & \leq E \left[\tilde{v}(\theta, y_n Y_\theta^{\rho, D}) + \int_{t_n}^\theta y_n Y_s^{\rho, D} \left(\alpha - \beta + \left(\beta - \int_C \rho_s(z) z \pi(dz) \right)_+ \right) ds \right] - \zeta \end{aligned} \quad (6.36)$$

where $\delta_n := \psi(t_n, y_n) - \tilde{v}(t_n, y_n)$.

Since $\delta_n = \psi(t_n, y_n) - \psi(\bar{t}, \bar{y}) + \tilde{v}_*(\bar{t}, \bar{y}) - \tilde{v}(t_n, y_n)$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\delta_n \geq -\frac{\zeta}{2}$. Inequality (6.36) implies

$$\begin{aligned} & \tilde{v}(t_n, y_n) \\ & \leq \inf_{Y^{\rho, D} \in \mathcal{Y}^0(t_n)} E \left[\tilde{v}(\theta, y_n Y_\theta^{\rho, D}) + \int_{t_n}^\theta y_n Y_s^{\rho, D} \left(\alpha - \beta + \left(\beta - \int_C \rho_s(z) z \pi(dz) \right)_+ \right) ds \right] - \frac{\zeta}{2}, \end{aligned}$$

which is a contradiction with the dynamic programming principle and so we conclude that the dual value function \tilde{v} is a viscosity super-solution. \square

6.3 Proof of Theorem 5.1

For $\epsilon, \lambda, \delta, \zeta > 0$, we define $\Phi : [0, T] \times (0, \infty) \times (0, \infty) \longrightarrow \mathbb{R} \cup \{-\infty\}$ as

$$\begin{aligned} \Phi(t, y_1, y_2) &:= \tilde{v}_1^*(t, y_1) - \tilde{v}_{2*}(t, y_2) - \frac{1}{\epsilon}(y_1 - y_2)^2 \\ &\quad - \delta \exp(\lambda(T - t)) \left(y_1^{\gamma+1} + y_2^{\gamma+1} \right) - \zeta \left(\frac{1}{y_1^{\gamma+1}} + \frac{1}{y_2^{\gamma+1}} \right). \end{aligned}$$

Since $\tilde{v}_1^*, \tilde{v}_{2*} \in D_\gamma([0, T] \times (0, \infty))$, there exists $(t^{*\epsilon, \delta, \lambda, \zeta}, x^{*\epsilon, \delta, \lambda, \zeta}, y^{*\epsilon, \delta, \lambda, \zeta}) \in [0, T] \times (0, \infty) \times (0, \infty)$ which maximizes Φ . By using the inequality

$$\begin{aligned} & 2\Phi(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \\ & \geq \Phi(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) + \Phi(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}), \end{aligned}$$

we have

$$\begin{aligned} \frac{2}{\epsilon} (y_1^{*\epsilon, \delta, \lambda, \zeta} - y_2^{*\epsilon, \delta, \lambda, \zeta})^2 &\leq \tilde{v}_1^*(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) - \tilde{v}_1^*(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \\ &\quad + \tilde{v}_{2*}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) - \tilde{v}_{2*}(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}). \end{aligned}$$

since $\tilde{v}_1^*, \tilde{v}_{2*} \in D_\gamma([0, T] \times (0, \infty))$, we have

$$\frac{2}{\epsilon} |y_1^{*\epsilon, \delta, \lambda, \zeta} - y_2^{*\epsilon, \delta, \lambda, \zeta}| \leq C \left(1 + \frac{1}{(y_1^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1}} + \frac{1}{(y_2^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1}} \right). \quad (6.37)$$

Using the inequality $\Phi(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \geq \Phi(T, 1, 1)$ and since $\tilde{v}_1^*, \tilde{v}_{2*} \in D_\gamma([0, T] \times (0, \infty))$, we have

$$\begin{aligned} & \delta \left((y_1^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1} + (y_2^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1} \right) \\ & + \zeta \left(\frac{1}{(y_1^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1}} + \frac{1}{(y_2^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1}} \right) \\ & \leq C^{\delta, \zeta} \left(1 + y_1^{*\epsilon, \delta, \lambda, \zeta} + y_2^{*\epsilon, \delta, \lambda, \zeta} + \frac{1}{(y_1^{*\epsilon, \delta, \lambda, \zeta})^\gamma} + \frac{1}{(y_2^{*\epsilon, \delta, \lambda, \zeta})^\gamma} \right), \end{aligned} \quad (6.38)$$

where $C^{\delta,\zeta}$ is a constant depending only on δ and ζ . Inequality (6.38) implies either

$$\delta(y_1^{*\epsilon,\delta,\lambda,\zeta})^{\gamma+1} + \zeta \frac{1}{(y_1^{*\epsilon,\delta,\lambda,\zeta})^{\gamma+1}} \leq C^{\delta,\zeta} \left(1 + y_1^{*\epsilon,\delta,\lambda,\zeta} + \frac{1}{(y_1^{*\epsilon,\delta,\lambda,\zeta})^\gamma} \right) \quad (6.39)$$

or

$$\delta(y_2^{*\epsilon,\delta,\lambda,\zeta})^{\gamma+1} + \tilde{\zeta} \frac{1}{(y_2^{*\epsilon,\delta,\lambda,\zeta})^{\gamma+1}} \leq C^{\delta,\zeta} \left(1 + y_2^{*\epsilon,\delta,\lambda,\zeta} + \frac{1}{(y_2^{*\epsilon,\delta,\lambda,\zeta})^\gamma} \right).$$

Assume the first case, then there exist $M_1^{\delta,\zeta}, M_2^{\delta,\zeta} > 0$ depending only on δ and ζ such that $M_1^{\delta,\zeta} \leq y_1^{*\epsilon,\delta,\lambda,\zeta} \leq M_2^{\delta,\zeta}$. Using Inequality (6.38), we obtain

$$\delta(y_2^{*\epsilon,\delta,\lambda,\zeta})^{\gamma+1} \leq C^{\delta,\zeta} \left(1 + y_1^{*\epsilon,\delta,\lambda,\zeta} + y_2^{*\epsilon,\delta,\lambda,\zeta} + \frac{1}{(y_1^{*\epsilon,\delta,\lambda,\zeta})^\gamma} + \frac{1}{(y_2^{*\epsilon,\delta,\lambda,\zeta})^\gamma} \right)$$

and

$$\zeta \frac{1}{(y_2^{*\epsilon,\delta,\lambda,\zeta})^{\gamma+1}} \leq C^{\delta,\zeta} \left(1 + y_1^{*\epsilon,\delta,\lambda,\zeta} + y_2^{*\epsilon,\delta,\lambda,\zeta} + \frac{1}{(y_1^{*\epsilon,\delta,\lambda,\zeta})^\gamma} + \frac{1}{(y_2^{*\epsilon,\delta,\lambda,\zeta})^\gamma} \right),$$

which implies that $M_1^{\delta,\zeta} \leq y_2^{*\epsilon,\delta,\lambda,\zeta} \leq M_2^{\delta,\zeta}$. Since $y_1^{*\epsilon,\delta,\lambda,\zeta}$ and $y_2^{*\epsilon,\delta,\lambda,\zeta}$ are bounded from below, inequality (6.37) implies

$$|y_1^{*\epsilon,\delta,\lambda,\zeta} - y_2^{*\epsilon,\delta,\lambda,\zeta}| \leq C_1 \epsilon, \quad (6.40)$$

where C_1 is a positive constant independent of ϵ . Using the boundedness of $y_1^{*\epsilon,\delta,\lambda,\zeta}$ and $y_2^{*\epsilon,\delta,\lambda,\zeta}$ and (6.40), along a subsequence $(t^{*\epsilon,\delta,\lambda,\zeta}, y_1^{*\epsilon,\delta,\lambda,\zeta}, y_2^{*\epsilon,\delta,\lambda,\zeta})$ converges when $\epsilon \rightarrow 0$. Let's denote $(t^{*\delta,\lambda,\zeta}, y^{*\delta,\lambda,\zeta})$ its limit.

From the definition of $t^{*\epsilon,\delta,\lambda,\zeta}$, two cases are possible:

★ Case 1: If the set $\{\epsilon > 0 : t^{*\epsilon,\delta,\lambda,\zeta} = T\}$ is not finite, then there exists a subsequence renamed $(t^{*\epsilon,\delta,\lambda,\zeta})_\epsilon$ such that $t^{*\epsilon,\delta,\lambda,\zeta} = T$. From inequality

$$\Phi(t, y, y) \leq \Phi(t^{*\epsilon,\delta,\lambda,\zeta}, y_1^{*\epsilon,\delta,\lambda,\zeta}, y_2^{*\epsilon,\delta,\lambda,\zeta})$$

and since Φ is upper semi-continuous, we deduce that

$$\begin{aligned} \Phi(t, y, y) &\leq \limsup_{\epsilon \rightarrow 0} \Phi(t^{*\epsilon,\delta,\lambda,\zeta}, y_1^{*\epsilon,\delta,\lambda,\zeta}, y_2^{*\epsilon,\delta,\lambda,\zeta}) \\ &\leq \Phi(T, y^{*\delta,\lambda,\zeta}, y^{*\delta,\lambda,\zeta}), \end{aligned}$$

which implies

$$\begin{aligned} &\tilde{v}_1^*(t, y) - \tilde{v}_{2*}(t, y) - 2\delta \exp(\lambda(T-t))y^{\gamma+1} - 2\zeta \frac{1}{y^{\gamma+1}} \\ &\leq \tilde{v}_1^*(T, y^{*\delta,\lambda,\zeta}) - \tilde{v}_{2*}(T, y^{*\delta,\lambda,\zeta}). \end{aligned}$$

Using inequality $\tilde{v}_1^*(T, y^{*\delta,\lambda,\zeta}) \leq \tilde{v}_{2*}(T, y^{*\delta,\lambda,\zeta})$ and sending $\lambda, \delta, \zeta \rightarrow 0^+$, we have

$$\tilde{u}^*(t, y) \leq \tilde{v}_*(t, y), \quad \text{for all } (t, y) \in [0, T] \times (0, \infty).$$

★ Case 2: If the set $\{\epsilon > 0 : t^{*\epsilon, \delta, \lambda, \zeta} = T\}$ is finite, then there exists a subsequence renamed $(t^{*\epsilon, \delta, \lambda, \zeta})_\epsilon$ such that $t^{*\epsilon, \delta, \lambda, \zeta} < T$. Our aim is to construct a regular function denoted $\tilde{\psi}_1$ (resp. $\tilde{\psi}_2$) satisfying inequality (4.5) (resp. (4.6)). We define ψ_1 and ψ_2 as follows

$$\begin{aligned}\psi_1(t, y) &:= \tilde{v}_{*2}(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) + \frac{1}{\epsilon}(y - y_2^{*\epsilon, \delta, \lambda, \zeta})^2 \\ &+ \delta \left(\exp(\lambda(T - t))y^{\gamma+1} + \exp(\lambda(T - t^{*\epsilon, \delta, \lambda, \zeta}))(y_2^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1} \right) \\ &+ \zeta \left(\frac{1}{y^{\gamma+1}} + \frac{1}{(y_2^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1}} \right) + \Phi(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}), \quad (t, y) \in [0, T] \times (0, \infty),\end{aligned}$$

and

$$\begin{aligned}\psi_2(t, y) &:= \tilde{v}_1^*(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) - \frac{1}{\epsilon}(y_1^{*\epsilon, \delta, \lambda, \zeta} - y)^2 \\ &- \delta \left(\exp(\lambda(T - t^{*\epsilon, \delta, \lambda, \zeta}))(y_1^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1} + \exp(\lambda(T - t))y^{\gamma+1} \right) \\ &- \zeta \left(\frac{1}{(y_1^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1}} + \frac{1}{y^{\gamma+1}} \right) - \Phi(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}), \quad (t, y) \in [0, T] \times (0, \infty).\end{aligned}$$

From inequalities (6.1) and (6.5), we have

$$\tilde{U}(y) - |K'|y \leq \tilde{v}(t, y) \leq \tilde{U}(y) + |K|y. \quad (6.41)$$

We define $\tilde{\psi}_1$ and $\tilde{\psi}_2$ as follows

$$\tilde{\psi}_1(t, y) = \begin{cases} 2\tilde{U}(y) & \text{for all } 0 < y \leq \underline{y}_1 \wedge \frac{M_1^{\delta, \zeta}}{2}, \\ \psi_1(t, y) & \text{for all } M_1^{\delta, \zeta} \leq y \leq M_2^{\delta, \zeta} \\ M_1 y & \text{for all } y \geq 2M_2^{\delta, \zeta}, \end{cases} \quad (6.42)$$

and

$$\tilde{\psi}_2(t, y) = \begin{cases} \frac{\tilde{U}(y)}{2} & \text{for all } 0 < y \leq \underline{y}_2 \wedge \frac{M_1^{\delta, \zeta}}{2}, \\ \psi_2(t, y) & \text{for all } M_1^{\delta, \zeta} \leq y \leq M_2^{\delta, \zeta} \\ M_2 y + M_2' & \text{for all } y \geq 2M_2^{\delta, \zeta}, \end{cases} \quad (6.43)$$

where $\underline{y}_1 = (\gamma|K|)^{-\frac{1}{\gamma+1}}$, M_1 satisfies

$$M_1 \geq \frac{(2M_2^{\delta, \zeta})^{-(\gamma+1)}}{\gamma} + |K| \vee \frac{\partial \psi_1(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta})}{\partial y}, \quad (6.44)$$

$\underline{y}_2 = (2\gamma|K'|)^{-\frac{1}{\gamma+1}}$, M_2 satisfies

$$\frac{\partial \psi_2(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta})}{\partial y} \leq M_2 \leq \tilde{v}_{*d}'(t^{*\epsilon, \delta, \lambda, \zeta}, 2M_2^{\delta, \zeta}), \quad (6.45)$$

where \tilde{v}_{*d}' is the right-hand derivative of \tilde{v}_{*2} with respect to the variable y and $M_2' = \tilde{v}_{*2}(t, 2M_2^{\delta, \zeta}) - 2M_2 M_2^{\delta, \zeta} - 1$. The choice of M_2 is possible since \tilde{v}_{*2} is convex and nonincreasing (see assumptions of the comparison theorem) and so

$$\tilde{v}_{*d}'(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \leq \tilde{v}_{*d}'(t^{*\epsilon, \delta, \lambda, \zeta}, 2M_2^{\delta, \zeta})$$

and from the optimality of $(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta})$, we have

$$\frac{\partial \psi_2(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta})}{\partial y} \leq \tilde{v}'_{2*d}(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}). \quad (6.46)$$

From the definition of \underline{y}_1 and inequalities (6.41), (6.44), we have

$$\begin{cases} \tilde{v}(t, y) \leq 2\tilde{U}(y) & \text{for all } 0 < y \leq \underline{y}_1 \\ \tilde{v}(t, y) \leq \tilde{U}(y) + |K|y \leq M_1 y & \text{for all } y \geq 2M_2^{\delta, \zeta} \end{cases}$$

and so one could obtain inequality (4.5) for $\tilde{\psi}_1$.

From the definition of \underline{y}_2 , inequalities (6.41), (6.46) and using the convexity of \tilde{v}_{2*d} , we have

$$\begin{cases} \tilde{v}(t, y) \geq \frac{\tilde{U}(y)}{2} & \text{for all } 0 < y \leq \underline{y}_2 \\ \tilde{v}(t, y) \geq M_2 y + M_2' & \text{for all } y \geq 2M_2^{\delta, \zeta} \end{cases}$$

and so one could obtain inequality (4.6) for $\tilde{\psi}_2$.

To prove the comparison theorem, we need to derive an equivalent formulation of viscosity solutions as in Soner [23] Lemma 2.1. For this, we show that the control ρ runs along a compact set and the lower semi-continuous envelope of H is continuous in its arguments which is the object of the next lemma. We denote by $\bar{\rho} := \frac{\beta}{\min_{1 \leq i \leq d} \delta_i \pi_i}$. We define Σ' by

$$\Sigma' = \left\{ \rho = (\rho_i)_{1 \leq i \leq d}, \ 0 \leq \rho_i \leq \frac{2M_2^{\delta, \zeta} + 1}{y_1^{*\epsilon, \delta, \lambda, \zeta}} \vee \frac{2M_2^{\delta, \zeta} + 1}{y_2^{*\epsilon, \delta, \lambda, \zeta}} \vee \bar{\rho} \right\} \quad (6.47)$$

and the Hamiltonian \tilde{H} by

$$\tilde{H} \left(t, y, \tilde{v}, \frac{\partial \tilde{v}}{\partial y} \right) = \inf_{\rho \in \Sigma'} \left\{ A^\rho \left(t, y, \tilde{v}, \frac{\partial \tilde{v}}{\partial y} \right) + y \left(\alpha - \beta + \left(\beta - \sum_{i=1}^d \rho_i \delta_i \pi_i \right)_+ \right) \right\}, \quad (6.48)$$

Lemma 6.1 *We assume that \tilde{v}_{2*} is convex and nonincreasing. Then, we have the following inequalities*

$$\begin{aligned} \min \left\{ \frac{\partial \psi_1}{\partial t}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) + \tilde{H} \left(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, \tilde{v}_1^*, \frac{\partial \psi_1}{\partial y} \right), \right. \\ \left. - \frac{\partial \psi_1}{\partial y}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) \right\} \geq 0, \end{aligned} \quad (6.49)$$

and

$$\begin{aligned} \min \left\{ \frac{\partial \psi_2}{\partial t}(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) + \tilde{H} \left(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}, \tilde{v}_{2*}, \frac{\partial \psi_2}{\partial y} \right), \right. \\ \left. - \frac{\partial \psi_2}{\partial y}(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \right\} \leq 0. \end{aligned} \quad (6.50)$$

Proof. If there exists $0 \leq i_0 \leq d$ such that $\rho_{i_0} \geq \bar{\rho}$, then $(\beta - \sum_{i=1}^d \rho_i \delta_i \pi_i)_+ = 0$. Let $\rho \in \Sigma$

be a fixed vector. If $\rho_i \geq \frac{2M_2^{\delta, \zeta}}{y_1^{*\epsilon, \delta, \lambda, \zeta}} \vee \bar{\rho}$, $0 \leq i \leq d$, then we have

$$\begin{aligned} F_1(\rho_i) &:= \tilde{\psi}_1(t^{*\epsilon, \delta, \lambda, \zeta}, \rho_i y_1^{*\epsilon, \delta, \lambda, \zeta}) - \tilde{\psi}_1(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) \\ &\quad - (\rho_i - 1) y_1^{*\epsilon, \delta, \lambda, \zeta} \frac{\partial \tilde{\psi}_1}{\partial y}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) \\ &= (M_1 - \frac{\partial \psi_1}{\partial y}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta})) \rho_i y_1^{*\epsilon, \delta, \lambda, \zeta} - \psi_1(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) \\ &\quad + y_1^{*\epsilon, \delta, \lambda, \zeta} \frac{\partial \psi_1}{\partial y}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}). \end{aligned}$$

From (6.44), we deduce that the function $\rho_i \rightarrow F_1(\rho_i)$ is non-decreasing and so

$$\begin{aligned} &\inf_{\rho \in \Sigma} \left\{ A^\rho \left(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, \tilde{\psi}_1, \frac{\partial \tilde{\psi}_1}{\partial y} \right) + y_1^{*\epsilon, \delta, \lambda, \zeta} \left(\alpha - \beta + (\beta - \sum_{i=1}^d \rho_i \delta_i \pi_i)_+ \right) \right\} \\ &= \inf_{\rho \in \Sigma'} \left\{ A^\rho \left(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, \tilde{\psi}_1, \frac{\partial \tilde{\psi}_1}{\partial y} \right) + y_1^{*\epsilon, \delta, \lambda, \zeta} \left(\alpha - \beta + (\beta - \sum_{i=1}^d \rho_i \delta_i \pi_i)_+ \right) \right\} \\ &:= \inf_{\rho \in \Sigma'} f_1(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, \rho) \\ &:= v_1^{opt}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}). \end{aligned} \tag{6.51}$$

The criterion of the optimization problem (6.51) is continuous with respect to ρ , Σ' is a compact and so there exists ρ_1^* solution of (6.51). We consider a sequence $(t_k, y_k)_k \in [0, T] \times (0, \infty)$ such that $(t_k, y_k) \rightarrow (t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta})$ when n goes to infinity. We denote by ρ_k the optimum i.e.

$$v_1^{opt}(t_k, y_k) = f_1(t_k, y_k, \rho_k). \tag{6.52}$$

Since $\rho_k \in \Sigma'$ which is compact, then along a subsequence denoted also by $(\rho_k)_k$, we have $\rho_k \rightarrow \bar{\rho}$. From the Taylor expansion formula and using the continuity of f_1 in her arguments, we have

$$\begin{aligned} f_1(t_k, y_k, \rho_k) &= f_1(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, \bar{\rho}) + o(1) \\ &\geq v_1^{opt}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) + o(1) \end{aligned} \tag{6.53}$$

From (6.52) and (6.53), we deduce that

$$v_1^{opt}(t_k, y_k) \geq v_1^{opt}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) + o(1). \tag{6.54}$$

To obtain the converse inequality, we have

$$\begin{aligned} v_1^{opt}(t_k, y_k) &\leq f_1(t_k, y_k, \rho_1^*) \\ &= f_1(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, \rho_1^*) + o(1) \\ &= v_1^{opt}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) + o(1), \end{aligned}$$

and so $\lim_{k \rightarrow \infty} v_1^{opt}(t_k, y_k) = v_1^{opt}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta})$. This proves that v_1^{opt} is continuous in $(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta})$ and so

$$H_* \left(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, \tilde{\psi}_1, \frac{\partial \tilde{\psi}_1}{\partial y} \right) = \tilde{H} \left(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, \tilde{\psi}_1, \frac{\partial \tilde{\psi}_1}{\partial y} \right). \quad (6.55)$$

From equality (6.42), the function $\tilde{v}_1 - \tilde{\psi}_1$ has a strict global minimum at $(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) \in [0, T) \times (0, \infty)$. Using the definition of viscosity supersolutions (see inequality (4.5)), equation (6.55) and (6.51), we obtain

$$\begin{aligned} & \min \left\{ \frac{\partial \tilde{\psi}_1}{\partial t}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) + \tilde{H} \left(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, \tilde{\psi}_1, \frac{\partial \tilde{\psi}_1}{\partial y} \right), \right. \\ & \left. - \frac{\partial \tilde{\psi}_1}{\partial y}(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) \right\} \geq 0, \end{aligned} \quad (6.56)$$

Let $\rho \in \Sigma$ be a fixed vector. If $\rho_i \geq \frac{2M_2^{\delta, \zeta}}{y_2^{*\epsilon, \delta, \lambda, \zeta}} \vee \bar{\rho}$, then we have

$$\begin{aligned} & F_2(\rho_i) \\ &:= \tilde{\psi}_2(t^{*\epsilon, \delta, \lambda, \zeta}, \rho_i y_2^{*\epsilon, \delta, \lambda, \zeta}) - \tilde{\psi}_2(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) - (\rho_i - 1) y_2^{*\epsilon, \delta, \lambda, \zeta} \frac{\partial \tilde{\psi}_2}{\partial y}(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \\ &= (M_2 - \frac{\partial \psi_2}{\partial y}(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta})) \rho_i y_2^{*\epsilon, \delta, \lambda, \zeta} - \psi_2(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \\ &+ y_2^{*\epsilon, \delta, \lambda, \zeta} \frac{\partial \psi_2}{\partial y}(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}). \end{aligned}$$

From the definition of $\tilde{\psi}_2$ and inequality (6.45), we deduce that the function $\rho_i \rightarrow F_2(\rho_i)$ is non-decreasing and so

$$\begin{aligned} & \inf_{\rho \in \Sigma} \left\{ A^\rho \left(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}, \tilde{\psi}_2, \frac{\partial \tilde{\psi}_2}{\partial y} \right) + y_2^{*\epsilon, \delta, \lambda, \zeta} \left(\alpha - \beta + \left(\beta - \sum_{i=1}^d \rho_i \delta_i \pi_i \right)_+ \right) \right\} \\ &= \inf_{\rho \in \Sigma'} \left\{ A^\rho \left(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}, \tilde{\psi}_2, \frac{\partial \tilde{\psi}_2}{\partial y} \right) + y_2^{*\epsilon, \delta, \lambda, \zeta} \left(\alpha - \beta + \left(\beta - \sum_{i=1}^d \rho_i \delta_i \pi_i \right)_+ \right) \right\}, \end{aligned} \quad (6.57)$$

From equality (6.43), $\tilde{v}_2 - \tilde{\psi}_2$ has a strict global maximum at $(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \in [0, T) \times (0, \infty)$. Using the definition of viscosity sub-solutions (see inequality (4.6)) and (6.57), we obtain

$$\begin{aligned} & \min \left\{ \frac{\partial \tilde{\psi}_2}{\partial t}(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) + \tilde{H} \left(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}, \tilde{\psi}_2, \frac{\partial \tilde{\psi}_2}{\partial y} \right), \right. \\ & \left. - \frac{\partial \tilde{\psi}_2}{\partial y}(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \right\} \geq 0. \end{aligned} \quad (6.58)$$

From inequalities (6.56) and (6.58), using the fact the control ρ runs along a compact set and repeating arguments of Soner [23] Lemma 2.1, we easily obtain an equivalent formulation of viscosity solutions given by inequalities (6.49) and (6.50). \square

We come back to the proof of the comparison theorem. Remarking that $\min\{d, e\} - \min\{f, g\} \geq 0$ implies either $d - f \geq 0$ or $e - g \geq 0$, inequalities (6.49) and (6.50) imply

$$-\delta\lambda \exp(\lambda(T - t^{*\epsilon, \delta, \lambda, \zeta})) \left((y_1^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1} + (y_2^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1} \right) + T_1 - T_2 \geq 0, \quad (6.59)$$

or

$$\begin{aligned} & -\delta \exp\left(\lambda(T - t^{*\epsilon, \delta, \lambda, \zeta})\right) \left((y_1^{*\epsilon, \delta, \lambda, \zeta})^\gamma + (y_2^{*\epsilon, \delta, \lambda, \zeta})^\gamma \right) \\ & + \zeta \left(\frac{1}{(y_1^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+2}} + \frac{1}{(y_2^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+2}} \right) \geq 0, \end{aligned} \quad (6.60)$$

where

$$\begin{aligned} T_1 &:= \inf_{\rho \in \Sigma'} \left\{ \sum_{i=1}^d \pi_i \left(\tilde{v}_1^*(t^{*\epsilon, \delta, \lambda, \zeta}, \rho_i y_1^{*\epsilon, \delta, \lambda, \zeta}) - \tilde{v}_1^*(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}) \right. \right. \\ & - (\rho_i - 1) y_1^{*\epsilon, \delta, \lambda, \zeta} \left(\frac{2}{\epsilon} (y_1^{*\epsilon, \delta, \lambda, \zeta} - y_2^{*\epsilon, \delta, \lambda, \zeta}) + \delta(\gamma + 1) \exp(\lambda(T - t^{*\epsilon, \delta, \lambda, \zeta})) (y_1^{*\epsilon, \delta, \lambda, \zeta})^\gamma \right. \\ & \left. \left. - \zeta(\gamma + 1) \frac{1}{(y_1^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+2}} \right) \right) + y_1^{*\epsilon, \delta, \lambda, \zeta} \left(\alpha - \beta + \left(\beta - \sum_{i=1}^d \rho_i \delta_i \pi_i \right)_+ \right) \right\}, \\ T_2 &:= \inf_{\rho \in \Sigma'} \left\{ \sum_{i=1}^d \pi_i \left(\tilde{v}_{2*}(t^{*\epsilon, \delta, \lambda, \zeta}, \rho_i y_2^{*\epsilon, \delta, \lambda, \zeta}) - \tilde{v}_{2*}(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \right. \right. \\ & - (\rho_i - 1) y_2^{*\epsilon, \delta, \lambda, \zeta} \left(\frac{2}{\epsilon} (y_1^{*\epsilon, \delta, \lambda, \zeta} - y_2^{*\epsilon, \delta, \lambda, \zeta}) - \delta(\gamma + 1) \exp(\lambda(T - t^{*\epsilon, \delta, \lambda, \zeta})) (y_2^{*\epsilon, \delta, \lambda, \zeta})^\gamma \right. \\ & \left. \left. + \zeta(\gamma + 1) \frac{1}{(y_2^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+2}} \right) \right) + y_2^{*\epsilon, \delta, \lambda, \zeta} \left(\alpha - \beta + \left(\beta - \sum_{i=1}^d \rho_i \delta_i \pi_i \right)_+ \right) \right\} \end{aligned} \quad (6.61)$$

and Σ' is defined in (6.47). From Lemma 5.1 \tilde{v}_{2*} is continuous w.r.t the state variable (\tilde{v}_{2*} is convex on $(0, \infty)$) and so the criterion of the optimization problem (6.61) is continuous w.r.t ρ . The set Σ' is compact and since $\lim_{y \rightarrow 0} v_{2*}(t, y) = \infty$, there exists a solution denoted by $\rho^{*\epsilon, \delta, \lambda, \zeta}$, to the optimization problem (6.61) satisfying $\rho_i^{*\epsilon, \delta, \lambda, \zeta} > 0$, for all $1 \leq i \leq d$.

We define f as follows:

$$\begin{aligned} f(\rho) &:= \sum_{i=1}^d \pi_i \left(\tilde{v}_{2*}(t^{*\epsilon, \delta, \lambda, \zeta}, \rho_i y_2^{*\epsilon, \delta, \lambda, \zeta}) - \tilde{v}_{2*}(t^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \right. \\ & - (\rho_i - 1) y_2^{*\epsilon, \delta, \lambda, \zeta} \left(\frac{2}{\epsilon} (y_1^{*\epsilon, \delta, \lambda, \zeta} - y_2^{*\epsilon, \delta, \lambda, \zeta}) - \delta(\gamma + 1) \exp(\lambda(T - t^{*\epsilon, \delta, \lambda, \zeta})) (y_2^{*\epsilon, \delta, \lambda, \zeta})^\gamma \right. \\ & \left. \left. + \zeta(\gamma + 1) \frac{1}{(y_2^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+2}} \right) \right) + y_2^{*\epsilon, \delta, \lambda, \zeta} \left(\alpha - \beta + \left(\beta - \sum_{i=1}^d \rho_i \delta_i \pi_i \right)_+ \right), \end{aligned}$$

From the definition of f , we have

$$\begin{aligned} T_2 = f(\rho^{*\epsilon, \delta, \lambda, \zeta}) &\leq f(\mathbf{1}) = y_2^{*\epsilon, \delta, \lambda, \zeta} (\alpha - \beta + (\beta - \sum_{i=1}^d \delta_i \pi_i)_+) \\ &\leq C^{\delta, \zeta}, \end{aligned} \quad (6.62)$$

where $\mathbf{1}$ denotes a \mathbb{R}^d -valued vector with all components equal to 1. Since

$$\Phi(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \geq \Phi(t^{*\epsilon, \delta, \lambda, \zeta}, \rho_i^{*\epsilon, \delta, \lambda, \zeta} y_1^{*\epsilon, \delta, \lambda, \zeta}, \rho_i^{*\epsilon, \delta, \lambda, \zeta} y_2^{*\epsilon, \delta, \lambda, \zeta})$$

for all $1 \leq i \leq d$, inequality (6.59) implies

$$- \delta \lambda \left(\exp(\lambda(T - t^{*\epsilon, \delta, \lambda, \zeta}))(y_1^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1} + \exp(\lambda(T - t^{*\epsilon, \delta, \lambda, \zeta}))(y_2^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1} \right) \geq T_2 - T_1 \geq T^{\rho^{*\epsilon, \delta, \lambda, \zeta}},$$

where

$$\begin{aligned} T^{\rho} &:= \left\{ \sum_{i=1}^d \pi_i \left((2(\rho_i - 1) + 1 - \rho_i^2) \frac{(y_1^{*\epsilon, \delta, \lambda, \zeta} - y_2^{*\epsilon, \delta, \lambda, \zeta})^2}{\epsilon} \right. \right. \\ &\quad + \delta((\gamma + 1)(\rho_i - 1) - \rho_i^{\gamma+1} + 1) \left(\exp(\lambda(T - t^{*\epsilon, \delta, \lambda, \zeta}))(y_1^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1} \right. \\ &\quad + \exp(\lambda(T - t^{*\epsilon, \delta, \lambda, \zeta}))(y_2^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1} \Big) \\ &\quad + \zeta \left((\gamma + 1)(1 - \rho_i) - \frac{1}{\rho_i^{\gamma+1}} + 1 \right) \left(\frac{1}{(y_1^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1}} + \frac{1}{(y_2^{*\epsilon, \delta, \lambda, \zeta})^{\gamma+1}} \right) \Big) \\ &\quad \left. + (y_2^{*\epsilon, \delta, \lambda, \zeta} - y_1^{*\epsilon, \delta, \lambda, \zeta})(\alpha - \beta + (\beta - \sum_{i=1}^d \rho_i \delta_i \pi_i)_+) \right\}. \end{aligned}$$

Sending $\epsilon \rightarrow 0^+$, we obtain

$$\begin{aligned} &- 2\delta \lambda \exp(\lambda(T - t^{*\delta, \lambda, \zeta}))(y^{*\delta, \lambda, \zeta})^{\gamma+1} \\ &\geq \sum_{i=1}^d \pi_i \left(2\delta \left((\gamma + 1)(\rho_i^{*\delta, \lambda, \zeta} - 1) + (\rho_i^{*\delta, \lambda, \zeta})^{\gamma+1} - 1 \right) \exp(\lambda(T - t^{*\delta, \lambda, \zeta}))(y^{*\delta, \lambda, \zeta})^{\gamma+1} \right. \\ &\quad \left. + 2\zeta \left((\gamma + 1)(1 - \rho_i^{*\delta, \lambda, \zeta}) - \frac{1}{(\rho_i^{*\delta, \lambda, \zeta})^{\gamma+1}} + 1 \right) \frac{1}{(y^{*\delta, \lambda, \zeta})^{\gamma+1}} \right). \end{aligned} \quad (6.63)$$

Using the boundedness of $y^{*\delta, \lambda, \zeta}$ and $\rho^{*\delta, \lambda, \zeta} \in \Sigma'$, along a subsequence $(y^{*\delta, \lambda, \zeta}, \rho^{*\delta, \lambda, \zeta})$ converges when $\lambda \rightarrow \infty$. Let's denote $(y^{*\delta, \zeta}, \rho^{*\delta, \zeta})$ its limit. Sending $\lambda \rightarrow \infty$ in inequality (6.63), we obtain

$$\begin{aligned} &\sum_{i=1}^d \pi_i \left(2\delta \left((\gamma + 1)(\rho_i^{*\delta, \zeta} - 1) + (\rho_i^{*\delta, \zeta})^{\gamma+1} - 1 \right) \exp(\lambda(T - t^{*\delta, \zeta}))(y^{*\delta, \zeta})^{\gamma+1} \right. \\ &\quad \left. + 2\zeta \left((\gamma + 1)(1 - \rho_i^{*\delta, \zeta}) - \frac{1}{(\rho_i^{*\delta, \zeta})^{\gamma+1}} + 1 \right) \frac{1}{y^{*\delta, \zeta}} \right) = -\infty \end{aligned}$$

which implies, there exists i_0 , $1 \leq i_0 \leq d$ such that $\rho_{i_0}^{*\delta, \zeta} = 0$. Sending $\epsilon \longrightarrow 0$ and $\lambda \longrightarrow \infty$ in inequality (6.62), we obtain $f(\rho^{*\delta, \zeta}) = \infty \leq C^{\delta, \zeta}$ which is false.

Sending $\epsilon \longrightarrow \infty$ in inequality (6.60), we have

$$-\delta \exp \left(\lambda(T - t^{*\delta, \lambda, \zeta}) \right) (y^{*\delta, \lambda, \zeta}) + \zeta \frac{1}{(y^{*\delta, \lambda, \zeta})^{\gamma+2}} \geq 0,$$

which implies

$$\frac{\delta \exp \left(\lambda(T - t^{*\delta, \lambda, \zeta}) \right)}{\zeta} \leq \frac{1}{(M_1^{\delta, \zeta})^{2(\gamma+1)}}. \quad (6.64)$$

Using the boundedness of $t^{*\delta, \lambda, \zeta}$, along a subsequence $t^{*\delta, \lambda, \zeta}$ converges when $\lambda \longrightarrow \infty$. From inequality (6.64), we have necessarily $t^{*\delta, \zeta} = T$.

From inequality

$$\Phi(t, y, y) \leq \Phi(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta})$$

and since Φ is upper semi-continuous, we deduce that

$$\begin{aligned} \Phi(t, y, y) &\leq \limsup_{\lambda \rightarrow \infty} \limsup_{\epsilon \rightarrow 0} \Phi(t^{*\epsilon, \delta, \lambda, \zeta}, y_1^{*\epsilon, \delta, \lambda, \zeta}, y_2^{*\epsilon, \delta, \lambda, \zeta}) \\ &\leq \Phi(T, y^{\delta, \zeta}, y^{\delta, \zeta}) \\ &\leq \tilde{v}_1^*(T, y^{\delta, \zeta}) - \tilde{v}_{2*}(T, y^{\delta, \zeta}) \leq 0. \end{aligned}$$

Sending $\delta, \zeta \longrightarrow 0^+$, we obtain

$$\tilde{v}_1^*(t, y) \leq \tilde{v}_{2*}(t, y), \text{ for all } (t, y) \in [0, T] \times (0, \infty).$$

and so Theorem 5.1 is proved. \square

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